DIFFERENTIABLE MANIFOLDS II: HOMEWORK 1

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(1) It is possible to present a vector bundle by providing only the local trivializations and transition maps between them, without positing the existence of a total space *a priori*.

Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of a smooth manifold X. For each ordered pair of indices (α, β), assume given $A_{\alpha\beta}$ a smooth map

$$A_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$$

Assume that

$$A_{\alpha\alpha}(x) = I$$

for all $x \in U_\alpha$, where I denotes the identity matrix in $\mathrm{GL}(k, \mathbb{R})$; also assume the cocycle condition

$$A_{\beta\gamma}(x) \cdot A_{\alpha\beta}(x) = A_{\alpha\gamma}(x)$$

for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Define a space E as the identification space

$$E = \bigcup_{\alpha \in A} \{\alpha\} \times U_\alpha \times \mathbb{R}^k / \sim$$

Where $\sim$ is the relation $(\alpha, x, v) \sim (\beta, y, w)$ iff $x = y$ as elements of X (meaning in particular that U_α and U_β have nonempty intersection) and $w = A_{\alpha\beta}(x)v$.

(a) Prove that $\sim$ is actually an equivalence relation.

(b) Show that E is a smooth manifold and construct a smooth map $\pi : E \to X$ making E into vector bundle over X.

(2) Use the preceding problem to give a construction of the tangent bundle "from scratch." Assume that X has a covering by open sets $\{U_\alpha\}_{\alpha \in A}$, each with a coordinate chart $\psi_\alpha : U_\alpha \to V_\alpha$, where $V_\alpha \subset \mathbb{R}^n$ is an open set.

(a) Write a precise expression for what should be the transition matrix

$$A_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$$

of the tangent bundle, in terms of the maps ψ_α. (Think about how vector fields transform under changes of coordinates.)

(b) Why is the cocycle condition satisfied?

(3) Let $\epsilon : S^1 \times \mathbb{R} \to S^1$ denote the trivial rank 1 bundle over S^1. Let $\mu : E \to S^1$ denote the Möbius bundle constructed in the lecture.

(a) Show that sections of ϵ correspond to periodic functions:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(\theta + 2\pi) = f(\theta)$$

(b) Show that sections of μ correspond to antiperiodic functions:

$$g : \mathbb{R} \to \mathbb{R}, \quad g(\theta + 2\pi) = -g(\theta)$$

(c) Show that any section of μ must vanish at some point. (That is, for any section s, there is $x \in S^1$ such that $s(x) = 0 \in \mu^{-1}(x)$).

(d) Deduce that ϵ and μ are not isomorphic vector bundles over S^1.

(4) The real projective space $\mathbb{RP}^n$ is defined as the set of lines through the origin in $\mathbb{R}^{n+1}$:

$$\mathbb{RP}^n = \{\ell \mid \ell \subset \mathbb{R}^{n+1} \text{ is a one-dimensional subspace}\}$$

This set may be given a topology and a manifold structure. The idea is that if two lines are "close" then one is the graph of a linear function on the other.

For a fixed $\ell_0 \in \mathbb{RP}^n$, let $\ell_0^\perp$ denote the orthogonal complement with respect to the standard dot product, and let $\mathrm{Hom}(\ell_0, \ell_0^\perp)$ denote the space of linear maps between these vector spaces. Define a local parametrization (the inverse of a local coordinate chart) by

$$\mathrm{Hom}(\ell_0, \ell_0^\perp) \to \mathbb{RP}^n, \quad f \mapsto \{v + f(v) | v \in \ell_0\}$$

Show that these parametrizations are injective, and that there is a smooth manifold structure on $\mathbb{RP}^n$ such that the inverses of the parametrizations are charts.

There is a natural line bundle on $\mathbb{RP}^n$ coming from the way it is constructed. Define $L \subset \mathbb{RP}^n \times \mathbb{R}^{n+1}$

$$L = \{(\ell, v) \mid v \in \ell\}$$

That is, L is the "incidence correspondence," the set of pairs of a line and a vector in that line. Show that L is a smooth submanifold of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$. Let $\tau : L \to \mathbb{RP}^n$ denote the projection onto the first factor. Show that τ is a rank 1 vector bundle. It is known as the *tautological line bundle over projective space.*

Some optional fun things to do with this:

(a) Show that when $n = 1$, $\mathbb{RP}^1$ is diffeomorphic to S^1, and τ is isomorphic to the Möbius bundle μ.
(b) Let $v_0 \in \mathbb{R}^{n+1}$ be a vector. Show that, by orthogonally projecting v_0 onto all lines, we may construct a section of τ. On what locus does this section vanish?
(c) There is another projection $\pi : L \to \mathbb{R}^{n+1}$. What are the fibers of this map?
(d) Compare this construction of projective space with another one you might know: the quotient of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ by the antipodal map $x \mapsto -x$.