A "simple" transversality theorem

In previous lecture, we saw how Fredholm topology could lead to smooth spaces of solutions to the J-holomorphic curve equation, provided that 0 is a regular value of the map $\bar\partial_J : Map(S,M) \to \Omega^{0,1}(S, u^*TM)$

Now there is no a priori reason for this to be true for any given J. But we can argue that, if the curves in question have a nice geometric property (somewhere injective) then by perturbing J slightly, we can make 0 a regular value.

Terminology: If 0 is a regular value of $\bar\partial_J$, then we say the solution set $\bar\partial_J^{-1}(0) = \{ u \in Map(S,M) \mid \bar\partial_J u = 0 \}$ is transversally cut out or regular, and we also say that J is regular.

In bundle formulation $\begin{array}{c} \mathcal{E} \\ \downarrow \\ \mathcal{B} \end{array} \circlearrowleft \bar\partial_J$, this means $\bar\partial_J \pitchfork$ 0-section

Sort of conclusion we are looking for: as $J \in \mathcal{J}$ varies through tame a.c.s.'s, the condition of being <u>regular</u> is <u>generic</u> or more precisely, defines a set of second Baire category (countable intersection of open dense sets)

That sort of conclusion reminds us of Sard-Smale theorem.

We consider applying Sard-Smale to the "parametrized moduli space"

$$\begin{array}{ccc} \mathcal{M} = \{ (u,J) \in \mathrm{Map}(S,M)\times\mathcal{J} & | & \bar\partial_J u = 0 \} \\ \downarrow & & \\ \mathcal{J} \ni J & & \end{array}$$

$$(u,J) \mapsto J$$

Of course, Sard-Smale only applies to Banach manifolds
→ Take maps of class L^p_k $(kp>2)$

→ For $\mathcal{J}$, at least 2 options:
(i) use $\mathcal{J}^\ell$ = a.c.s. of class C^ℓ
(note that solutions are then only C^ℓ)
(ii) Use a smaller space of Js which is a Banach manifold (Floer's C^ϵ-topology)

In applying Sard-Smale to $\begin{array}{c}\mathcal{M}\\ \pi\downarrow \\ \mathcal{J}\end{array}$ (Banach completion implied)

we must show that: (a) $\mathcal{M}$ is a Banach manifold
(b) π is a Fredholm map.

The map that defines $\mathcal{M}$ is

$$\bar\partial : \mathrm{Map}(S,M)\times\mathcal{J} \to \mathcal{E}$$

$$(u,J) \to \bar\partial_J u \in \underbrace{\Omega^{0,1}_J(S,u^*TM)}_{\text{This space depends on Both } u \text{ and } J.}$$

So $\mathcal{E}$ is really a bundle over $\mathrm{Map}(S,M)\times\mathcal{J}$

The section $\bar\partial : \mathrm{Map}(S,M)\times\mathcal{J} \to \mathcal{E}$ is not Fredholm, as the kernel will be infinite dimensional (corresponding to $T\mathcal{J}$)

But the map $\bar\partial\times\pi : \mathrm{Map}(S,M)\times\mathcal{J} \to \mathcal{E}\times\mathcal{J}$ is Fredholm, and its index is the same as $\mathrm{ind}(D_u\bar\partial_J)$ for fixed (u,J).

{For $\mathcal{M}$ to be a Banach manifold, we need $\bar\partial\times\pi$ to be transverse to the submanifold (0-section) $\times\,\mathcal{J}$ }

This we will prove <u>by hand</u> under restrictive hypotheses

After restricting to $\mathcal{M}$, there remains a map

$$\pi : \mathcal{M}\to\mathcal{J}$$

which, as a formal consequence, is Fredholm of index $\mathrm{ind}(D_u\bar\partial_J)$.

In linear algebra

$$\begin{array}{ccc} T_u\mathrm{Map}(S,M) & \xrightarrow{D_u\bar\partial_J \text{ - Fredholm}} & T_0\mathcal{E}_u \\ \times & & \times \\ T_J\mathcal{J} & \xrightarrow{\mathrm{Id}} & T_J\mathcal{J} \end{array}$$

$$T_u\mathcal{M} = (D_u\bar\partial_J\times \mathrm{Id})^{-1}(0\times T_J\mathcal{J})$$

The map $d\pi : T_u\mathcal{M}\to T_J\mathcal{J}$ is $(D_u\bar\partial_J\times\mathrm{Id})$ restricted to this subspace.

So this boils down to

<u>lemma</u>: if $D: X \to Y$ is Fredholm, and $\mathrm{Im}(D)$ is transverse to the closed subspace $V \subset Y$ (That is $\mathrm{Im}\, D + V = Y$) then

$$\tilde{D} = D\big|_{D^{-1}(V)} : D^{-1}(V) \to V$$ is Fredholm, with same kernel, dim coker, and index

<u>Proof</u>: Pick representatives for coker D and a complement for ker D:

$$D : \ker(D) \oplus C \to \mathrm{Im}\, D \oplus \mathrm{coker}\, D$$

V is transverse to $\mathrm{Im}\, D$, so projection of V on coker D is surjective

Now $\mathrm{Im}\, \tilde{D} = \mathrm{Im}\, D \cap V$, which has codimension in V equal to dim coker D

Now $D|_C : C \to \mathrm{Im}\, D$ is isomorphism.

And $D^{-1}(V) = \ker D \oplus \left((D|_C)^{-1} (\mathrm{Im}\, D \cap V) \right)$

So in fact $\ker(\tilde{D}) = \ker(D)$ $\boxed{///}$

In summary, applying Sard-Smale to $\pi : \mathcal{M} \to \mathcal{J}$, we find a set $\mathcal{J}^{reg}$ of second Baire category such that for $J \in \mathcal{J}^{reg}$ $\mathcal{M}(J) = \pi^{-1}(J)$ is a smooth manifold of the correct dimension $\mathrm{ind}(D_u \bar{\partial}_J)$.

__But__ it remains to set down geometric conditions that guarantee that $\bar\partial \times \pi$ is transverse to $0 \times \mathcal{J}$.

Multiply covered curves yield __counterexamples__ to this claim in general, so the hypotheses must be somewhat restrictive.

Suppose our curves are __somewhere injective__ which means $\exists\, p \in S$ s.t. $du_p \neq 0$ and $u^{-1}(u(p)) = \{p\}$

Then some neighborhood of p is embedded and disjoint from the rest of the image of u. The trick is that, in this neighborhood, the complex structure can vary arbitrarily, and it won't interact with other parts of the curve.

__Sketch proof__ that $\bar\partial : \mathrm{Map}(S,M) \times \mathcal{J} \to \mathcal{E}$ is transverse to zero in this case.

Suppose $\eta \in \operatorname{coker} D_{(u,J)}\bar\partial$ when (u,J) is a solution. We are done if we show $\eta = 0$.

Represent η as a section of $\Omega^{0,1}(S, u^*TM)$ which is L^2 orthogonal to the image of $D_{(u,J)}\bar\partial$

The image has two parts:

(I) $(D_{(u,J)}\bar\partial)\,\xi \qquad \xi \in T_u \mathrm{Map}(S,M)$: vary u, fix J

(II) $(D_{(u,J)}\bar\partial)\,Y \qquad Y \in T_J \mathcal{J}$: vary J fix u.

Correspondingly:

(I) $\langle \eta, (D_{(u,J)}\bar\partial)\xi \rangle = 0 \quad \forall \xi$

By duality theory η satsifies an equation of the form
$$(D_{(u,J)}\bar\partial)^* \eta = 0$$, which is an elliptic equation.

(II) $\langle \eta, (D_{(u,J)}\bar\partial) Y \rangle = 0 \quad \forall Y.$

Consider Y supported in the set where u is injective. There are many such vectors, and it is a linear algebraic fact that there are enough perturbations that all components of η must vanish in the open set where u is injective in order for II to be true.

So we conclude that η solves the equation $(D_{(u,J)}\bar\partial)^* \eta = 0$ and that η vanishes on a nonempty open set.

Now we apply a theorem of Aronszajn, which says that unique continuation holds for solutions of an equation like $(D_{(u,J)}\bar\partial)^* \eta = 0$.

So $\eta = 0$ on an open set $\Rightarrow$ $\eta \equiv 0$ everywhere. $\square$