Manifold Refreshment.

Recall M a smooth manifold,

Tangent bundle TM sections are vector fields $\mathfrak{X}(M)$

cotangent bundle T^*M sections are 1-forms $\Omega^1(M)$

wedge powers $\Lambda^k T^*M$ sections are k-forms $\Omega^k(M)$

We always work with C^∞ objects.

Given a vector field X on M, there is an associated ordinary differential equation

unknown: path $\gamma(t): \mathbb{R} \to M$ equation $\frac{d\gamma}{dt} = X(\gamma(t))$ $(*)$

ODE theory $\Rightarrow$ given any initial point $\gamma(0) = p$, we have existence and uniqueness for short time $\varepsilon(p)$

$$\exists! \; \gamma : [0, \varepsilon(p)] \to M \text{ satisfying } (*)$$

If solution curves exist for all points for all times, X is called a complete vector field. In this case, we get an isotopy $\rho_t : M \to M$, called the flow of X

$$\rho_t(x) = \begin{pmatrix} \text{take } \gamma(t), \text{ where } \gamma \text{ is the solution curve} \\ \text{with initial point } \gamma(0) = x \end{pmatrix}$$

- If M is compact, any vector field is complete.

- All of the above also holds if the vector field X has an explicit time dependence X_t. Then the ODE is $\frac{d\gamma}{dt} = X_t(\gamma(t))$. The flow is still denoted ρ_t.

Exterior derivative This is a first-order differential operator

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

It is determined by the following properties (which are typically most useful for computation as well)

- d is natural with respect to smooth maps $f: M \to N$: consider pull back $f^* : \Omega^k(N) \to \Omega^k(M)$ then $d(f^*\alpha) = f^*(d\alpha)$
- $d^2 : \Omega^k(M) \to \Omega^{k+1}(M) \to \Omega^{k+2}(M)$ is zero
- if $f \in \Omega^0(M)$ is a function, df is the differential of f: $df(X) = X \cdot f$ for any vector field X.
- if $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

In an arbitrary local coordinate system $(x_i)_{i=1}^n$ on M if $I \subset \{1, \dots, n\}$ is a subset of size k, we have a k form

$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}, \quad i_1 < i_2 < \dots < i_k \text{ being the elements of } I.$$

Any k-form has local expression $\alpha = \sum_{\substack{I \subset \{1,\dots,n\} \\ |I| = k}} f_I \, dx_I$

then $d\alpha = \sum_I df_I \wedge dx_I$, where $df_I = \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i$

Integral: Given a sufficiently smooth k-chain C in M (e.g. an oriented smooth submanifold of M), there is a well defined integral on k-forms

$$\int_C \alpha \quad \text{for } \alpha \in \Omega^k(M)$$

We have <u>Stokes' theorem</u>: $\int_C d\alpha = \int_{\partial C} \alpha$

where ∂C denotes the boundary of the chain C.

<u>Contraction</u>: let X be a vector field, α a k-form. There is a $(k-1)$ form $i_X\alpha$ called the contraction of X with α.
Think of α as an alternating k-multilinear form on the tangent spaces, and plug X into first input.

Derivation property: $i_X(\alpha\wedge\beta) = (i_X\alpha)\wedge\beta + (-1)^k \alpha\wedge(i_X\beta)$
if $\alpha\in\Omega^k(M)$ and $\beta\in\Omega^l(M)$.

<u>Homework</u>: prove this.

<u>Lie derivative</u>: Recall flow ρ_t of vector field X (time independent)

Then we can define $\mathcal{L}_X\alpha = \frac{d}{dt}(\rho_t^*\alpha)\Big|_{t=0}$

$$\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$$

This is an intrinsic notion of derivative that is <u>not</u> tensorial in X

<u>Cartan's Magic Formula</u> (or Cartan homotopy formula)

$$\mathcal{L}_X\alpha = d i_X\alpha + i_X d\alpha \qquad \text{for } \alpha\in\Omega^k(M)$$

<u>Derivative at other times</u>: $\frac{d}{dt}\rho_t^*\alpha = \rho_t^*\mathcal{L}_X\alpha$

If the vector field is time dependent X_t, and ρ_t is the flow,
We also have $\frac{d}{dt}\rho_t^*\alpha = \rho_t^*\mathcal{L}_{X_t}\alpha$

In $\mathcal{L}_{X_t}\alpha$, the vector field is first evaluated at time t.

If the k-form also has an explicit time dependence

$$\frac{d}{dt}\rho_t^* \alpha_t = \rho_t^*\left(\mathcal{L}_{X_t}\alpha_t + \frac{d\alpha_t}{dt}\right)$$

Symplectic manifolds A two form $\omega \in \Omega^2(M)$ is called symplectic if is nondegenerate (defines a linear symplectic form on each tangent space) and satisfies

$$d\omega = 0 \quad (\omega \text{ is closed})$$

- By nondegeracy, all the linear algebra carries over at the level of tangent spaces: can define isotropic, coisotropic, Lagrangian, symplectic submanifolds as those which satisfy the corresponding condition on tangent spaces. Also we have a Lagrangian grassmannian bundle

$$U(n)/O(n) \to \Lambda \to M$$

- The closedness condition $d\omega = 0$ is what really holds the geometry together, however, as we shall see. (Vaguely: it makes symplectic geometry "locally constant")

Symplectic isotopy: ρ_t isotopy generated by X_t vector field

When is ρ_t symplectic (for all t)? $\rho_t^*\omega = \omega$

$$0 \equiv \frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_t}\omega = \rho_t^*(d i_{X_t}\omega + i_{X_t} d\omega)$$

$$= \rho_t^*(d i_{X_t}\omega) \quad (\text{since } d\omega = 0)$$

$\iff$ $i_{X_t}\omega$ is closed for all t.

We call such X_t a symplectic vector field.

If $i_{X_t}\omega$ is not merely closed, but exact, X_t is called a Hamiltonian vector field.

<u>Hamiltonian vector field</u>: $f \in C^\infty(M)$ function $df \in \Omega^1(M)$ define vector field X_f by

$$\omega(\cdot, X_f) = -i_{X_f}\omega = df \quad \text{(uses nondegeneracy)}$$

f_t time-dependent family of functions: $\omega(\cdot, X_{f_t}) = df_t$

<u>Poisson bracket</u>: $\{f,g\} = \omega(X_f, X_g) \quad (f,g \in C^\infty(M))$

$$= \omega(X_f, X_g) = dg(X_f) = X_f \cdot g$$

$$= -\omega(X_g, X_f) = -df(X_g) = -X_g \cdot f$$

lemma: if X, Y are symplectic v.f. then $i_{[X,Y]}\omega = d(\omega(Y,X))$

since ω is closed

$$0 = d\omega(X,Y,Z) = X\cdot\omega(Y,Z) + Y\cdot\omega(Z,X) + Z\cdot\omega(X,Y)$$
$$- \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y)$$

since $i_X\omega$ is closed

$$0 = d(i_X\omega)(Y,Z) = Y\cdot i_X\omega(Z) - Z\, i_X\omega(Y) - i_X\omega([Y,Z])$$
$$0 = Y\cdot\omega(X,Z) - Z\cdot\omega(X,Y) - \omega(X,[Y,Z])$$
$$0 = Y\cdot\omega(Z,X) + Z\cdot\omega(X,Y) - \omega([Y,Z],X)$$

since $i_Y\omega$ closed

$$0 = X\cdot\omega(Z,Y) + Z\cdot\omega(Y,X) - \omega([X,Z],Y)$$
$$0 = X\cdot\omega(Y,Z) + Z\cdot\omega(X,Y) - \omega([Z,X],Y)$$

Combine $\Rightarrow$ $0 = -Z \cdot \omega(X,Y) - \omega([X,Y],Z)$
$\omega([X,Y],Z) = Z \cdot \omega(Y,X)$

Cor: if X, Y are symplectic v.f., $[X,Y]$ is hamiltonian v.f.

Prop: $[X_f, X_g] = X_{\{f,g\}}$

Proof $-i_{[X_f, X_g]}\omega = d(\omega(X_f, X_g)) = d\{f,g\}$

Prop Poisson bracket satisfies $\{f, gh\} = \{f,g\} \cdot h + g \cdot \{f,h\}$
and $\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0$
(Jacobi identity)

Proof

Derivation $\{f, gh\} = X_f(gh) = X_f(g) h + g X_f(h)$
$= \{f,g\} h + g\{f,h\}$

Jacobi: $0 = d\omega(X_1, X_2, X_3)$
$= X_1 \omega(X_2, X_3) + X_2 \omega(X_3, X_1) + X_3 \omega(X_1, X_2)$
$- \omega([X_1, X_2], X_3) - \omega([X_2, X_3], X_1) - \omega([X_3, X_1], X_2)$

$= X_1 \{f_2, f_3\} + X_2 \{f_3, f_1\} + X_3 \{f_1, f_2\}$
$- \omega(X_{\{f_1, f_2\}}, X_3) - \omega(X_{\{f_2, f_3\}}, X_1) - \omega(X_{\{f_3, f_1\}}, X_2)$
$= \{f_1 \{f_2 f_3\}\} + \{f_2 \{f_3 f_1\}\} + \{f_3 \{f_1 f_2\}\}$
$- \{\{f_1 f_2\} f_3\} - \{\{f_2 f_3\} f_1\} - \{\{f_3 f_1\} f_2\}$
$= 2 \cdot$ (Jacobi expression)

Point $d\omega = 0 \Rightarrow$ Jacobi identity.

Conservation of energy: If X_H is vector field of H, then $X_H \cdot H = 0$ i.e., H is constant along trajectories of X_H
proof $X_H \cdot H = \omega(X_H, X_H) = 0$ by skew symmetry.

Now we justify notion that symplectic geometry is "locally constant" in some senses.

<u>Moser theorem</u> let M be compact, and let ω_t be a family of symplectic forms such that $\frac{d\omega_t}{dt}$ is exact for all t.
Then there exists an isotopy $\rho_t : M \to M$ such that
$$\rho_t^* \omega_t = \omega_0 \quad \text{for all } t.$$

Investigate using Lie-Cartan calculus
<u>Want</u>: for vector field X_t generating ρ_t:

$$0 = \frac{d}{dt} \rho_t^* \omega_t = \rho_t^* \left(\mathcal{L}_{X_t} \omega_t + \frac{d\omega_t}{dt} \right)$$

$$0 = \mathcal{L}_{X_t} \omega_t + \frac{d\omega_t}{dt}$$

$$0 = d\, i_{X_t} \omega_t + i_{X_t} d\omega + \frac{d\omega_t}{dt} \quad \text{by Cartan Magic}$$

$$0 = d\, i_{X_t} \omega_t + \frac{d\omega_t}{dt} \quad \text{since } d\omega = 0$$

Now since $\frac{d\omega_t}{dt}$ is exact, we can choose β_t such that $\frac{d\omega_t}{dt} = d\beta_t$
(that β_t may be chosen to depend smoothly on t may be deduced from Hodge theory, for example)

Want $\quad 0 = d\, i_{X_t}\omega + d\beta_t$. It suffices to solve

$$0 = i_{X_t}\omega + \beta_t \qquad \text{(Moser's equation)}$$

This equation is uniquely solvable for X_t since ω is nondegenerate. This completes the proof.

- Moser's theorem shows that a 1-parameter family of symplectic manifolds with constant cohomology class of symplectic forms is <u>trivial</u>.

<u>Theorem</u> (Darboux–Weinstein) Let $N \subset M$ be a submanifold and let ω_0 and ω_1 be symplectic forms on a tubular neighborhood of N such that $\omega_0|_N = \omega_1|_N$. Then there possibly smaller tubular neighborhoods U_0 and U_1 and a diffeo $\phi : U_0 \to U_1$ such that $\phi|_N = \mathrm{id}_N$ and $\phi^*\omega_1 = \omega_0$.

<u>Proof</u> $\omega_1 - \omega_0$ is a form which vanishes when restricted to N. By the "relative Poincaré lemma" on a tubular neighborhood of N, there is a 1-form β on a tubular neighborhood such that $d\beta = \omega_1 - \omega_0$ and $\beta|_N = 0$

Consider the family of forms $\omega_t = \omega_0 + t\, d\beta$. There is a possibly smaller tubular neighborhood of N such that all these forms are symplectic there. In such a neighborhood, we can solve the Moser equation $\quad i_{X_t}\omega_t + \beta = 0$

Then the flow of X_t will satisfy $\rho_t^*\omega_t = \omega_0$

Note that X_t vanishes on N since β does.
so ρ_t is identity on N.

Cor (Darboux theorem) Let $p \in M$ be a point. Then there exists a neighborhood U of p and a neighborhood V of $0 \in \mathbb{R}^{2n}$ and a diffeomorphism $\phi : V \to U$ such that $\phi^* \omega_M = \omega_{std}$.

Proof Use linear standard form to construct $\phi : V \to U$
$(D\phi_p)^* \omega_{M,p} = \omega_{std,0}$ holds at the level of the tangent space to p. Then apply Darboux-Weinstein.

Another perspective on the Moser theorem:
What is a family of symplectic manifolds parametrized by a base B? (Assume all manifolds are diffeomorphic to a fixed compact manifold M)
One answer: As above, we could have a continuous/smooth map $B \to \{\text{Symplectic forms on } M\}$
Such families are slightly "wild"

Another "tamer" answer: A family of symplectic structures on M parametrized by B consists of
(a) a fibration $M \to \mathcal{M}$, $\downarrow \pi$, B
(assume it's differentiably locally trivial)
(b) a two-form $\Omega \in \Omega^2(\mathcal{M})$ such that
(i) $d\Omega = 0$ on $\mathcal{M}$
(ii) $\forall\, b \in B$ $\omega_b := \Omega|_{\pi^{-1}(b)}$ is a symplectic form on $M_b := \pi^{-1}(b)$.

<u>Theorem</u> Consider the case $B = I = [0,1]$. Any family of symplectic structures (in the "tamer" sense) over I is trivial. The trivialization is constructed canonically from Ω.

<u>Proof</u> Let $\begin{array}{c}(\mathcal{M}, \Omega)\\ \pi\downarrow \\ I\end{array}$ be such a family.

Let $t \in I = [0,1]$ be a coordinate, and let $X = \frac{\partial}{\partial t}$ be the standard vector field on I

<u>Claim</u> There is a unique vector field $\tilde{X}$ on $\mathcal{M}$ such that

(a) $i_{\tilde{X}}\Omega = 0$

(b) $D\pi_p(\tilde{X}_p) = X_{\pi(p)} \quad \forall\, p \in \mathcal{M}$

<u>Proof of claim</u>: Linear algebra. Ω is a 2-form on a $(2n+1)$-dim space. Since $\Omega|_{\pi^{-1}(t)}$ is always non degenerate, rank $\Omega \geq 2n$ since the rank is even, rank $\Omega = 2n$, an so Ω has a one-dimensional null space at each point. Thus there is a one-dimensional space of $\tilde{X}$ such that $i_{\tilde{X}}\Omega = 0$.
Such $\tilde{X}$ cannot be tangent to the fiber, since Ω is non degenerate on the fiber. Thus $i_{\tilde{X}}\Omega = 0 \Rightarrow D\pi(\tilde{X}) \neq 0$.
Since target is 1-dim, $D\pi(\tilde{X}) = \alpha X$ for some $\alpha \neq 0$.
Rescale $\tilde{X}$ if necessary to achieve property (b).
End proof of claim.

Now let ρ_s be the flow of $\tilde{X}$ on $\mathcal{M}$. Because $D\pi(\tilde{X}) = X$
ρ_s covers $\sigma_s := (\text{flow of } X \text{ on } I)$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\rho_s} & \mathcal{M} \\ \pi\downarrow & & \downarrow\pi \\ I & \xrightarrow{\sigma_s} & I \end{array}$$

Hence, ρ_s maps $M_t = \pi^{-1}(t)$ to $M_{t+s} = \pi^{-1}(t+s)$

(take obvious precautions about flow running off ends of interval)

(Note: If Fibers are not compact or have boundary, more care is needed)

Now note that ρ_s preserves Ω!

$$\mathcal{L}_{\tilde{X}}\Omega = d i_{\tilde{X}}\Omega + i_{\tilde{X}} d\Omega \qquad \text{Cartan magic}$$
$$= d i_{\tilde{X}}\Omega \qquad \text{since } d\Omega = 0$$
$$= 0 \qquad \text{since } i_{\tilde{X}}\Omega = 0$$

So $\rho_s^*\Omega = \Omega$

$$\begin{tikzcd} M_0 = \pi^{-1}(0) \arrow[r, hook, "i_0"] \arrow[d, "\rho_s|_{M_0}"'] & \mathcal{M} \arrow[d, "\rho_s"] \\ M_s = \pi^{-1}(s) \arrow[r, hook, "i_s"] & \mathcal{M} \end{tikzcd}$$

$i_0^*\rho_s^*\Omega = i_0^*\Omega = \omega_0$ on M_0

$(\rho_s|_{M_0})^* i_s^*\Omega = (\rho_s|_{M_0})^*\omega_s$

$\Longrightarrow \quad \omega_0 = (\rho_s|_{M_0})^*\omega_s.$

Conclude $\rho_s|_{M_0} : (M_0, \omega_0) \to (M_s, \omega_s)$ is a symplectic diffeomorphism for all $s \in [0,1]$ $\blacksquare$

Comparison with first proof:

Given: ω_t family such that $\frac{d\omega_t}{dt}$ is exact

What we need is to say that we can convert these data into a symplectic fibration (i.e. family in the "fancier" sense)

Indeed, Define a two-form ω on $M \times I$ Naively:

For $(p,t) \in M \times I$ $\quad T_{(p,t)}(M \times I) = T_pM \times \mathbb{R}$

Let $\omega_{(p,t)}$ on $T_pM \times \mathbb{R}$ be $(\omega_t)_p$ on T_pM and zero on $\mathbb{R}$ factor.

Clearly ω restricted to $M \times \{t\}$ is ω_t

Then ω is not closed!

$$\underset{\uparrow}{d}\omega = \underset{\uparrow}{\overset{0}{\overset{\|}{d_M\omega_t}}} + \underset{\uparrow}{dt \wedge \frac{d\omega_t}{dt}} = dt \wedge \frac{d\omega_t}{dt}$$

ext. dr. on $M\times I$ ext. dr. on M 3-form on $M\times I$

Correction term: as before, solve $d_M\beta_t = \frac{d\omega_t}{dt}$ on M
β_t yields a 1-form β on $M\times I$.

Define $\Omega = \omega + dt\wedge\beta$

$$d\Omega = d\omega + d(dt\wedge\beta) = dt\wedge\frac{d\omega_t}{dt} - dt\wedge d\beta$$

$$\left(\text{Now } d\beta = d_M\beta_t + dt\wedge\frac{d\beta_t}{dt} = \frac{d\omega_t}{dt} + dt\wedge\frac{d\beta_t}{dt}\right)$$

$$d\Omega = dt\wedge\frac{d\omega_t}{dt} - dt\wedge\frac{d\omega_t}{dt} - \underbrace{dt\wedge dt}_{=0}\wedge\frac{d\beta_t}{dt} = 0$$

Lastly, Ω restricted to $M\times\{t\}$ is ω_t, since $dt\wedge\beta$ restricts trivially