Symplectic Refreshment

<u>Linear algebra</u>: let V be a real vector space

let $\omega : V \times V \to \mathbb{R}$ be a skew-symmetric bilinear form
$\omega(x,y) = -\omega(y,x)$

ω defines a map $\tilde{\omega} : V \to V^*$

$x \in V \longmapsto \tilde{\omega}(x) = \big(\text{the linear function } y \mapsto \omega(x,y)\big)$

<u>Def</u>: ω is <u>nondegenerate</u> if $\tilde{\omega}$ is an isomorphism.

In the linear algebra context, ω is called a <u>symplectic form</u>, if it is skew-symmetric and nondegenerate.

[In the manifold context we also require $d\omega = 0$, but we'll get there]

Examples $V = \mathbb{R}^{2n}$, basis $\{e_1, e_2, \dots, e_{2n}\}$ of V

dual basis $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2n}\}$ of V^*, meaning $\varepsilon_i(e_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Now, given two linear forms $\alpha_1, \alpha_2 : V \to \mathbb{R}$, we may construct skew-symmetric bilinear form $\alpha_1 \wedge \alpha_2$

$$(\alpha_1 \wedge \alpha_2)(v_1, v_2) = \alpha_1(v_1)\alpha_2(v_2) - \alpha_1(v_2)\alpha_2(v_1)$$

Let $\omega = \varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4 + \cdots + \varepsilon_{2n-1} \wedge \varepsilon_{2n}$

Thus $\omega(e_i, e_j) = \begin{cases} 1 & \text{if } i=2k-1,\ j=2k \\ -1 & \text{if } i=2k,\ j=2k-1 \\ 0 & \text{otherwise.} \end{cases}$

so $\tilde{\omega}(e_1) = \varepsilon_2 \quad \tilde{\omega}(e_2) = -\varepsilon_1$, etc

The matrix of $\tilde{\omega} : V \to V^*$ wrt. bases e_i ε_i is

$$\left[\begin{array}{c|c|c} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & \ddots \end{array}\right]$$

block-diagonal with 2×2 blocks $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

So $\tilde{\omega}$ is an isomorphism, and ω is symplectic

<u>Example</u>: let Q be any vector space, and consider

$V = Q \oplus Q^*$. Denote elements of $Q \oplus Q^*$ as $q \oplus p$ where $q \in Q$, $p \in Q^*$. Define a form

$$\omega(q_1 \oplus p_1, q_2 \oplus p_2) = p_2(q_1) - p_1(q_2)$$

evaluation of Q^* on Q.

All symplectic vector spaces of a given dimension are isomorphic, in particular, isomorphic to $(\mathbb{R}^{2n}, \omega = \varepsilon_1 \wedge \varepsilon_2 + \cdots + \varepsilon_{2n-1} \wedge \varepsilon_{2n})$

<u>Thm</u> Let ω be a symplectic form on V. Then V is of even dimension $2n$, and there is a basis $\{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$

such that $\begin{cases} \omega(a_i, a_j) = 0 = \omega(b_i, b_j) \\ \omega(a_i, b_j) = \delta_{ij} \end{cases}$

Equivalently, if $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_n, \beta_n\}$ is dual to $\{a_1, b_1, a_2, b_2, \dots, a_n, b_n\}$

Then $\omega = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 + \dots + \alpha_n \wedge \beta_n$.

<u>lemma</u> let V^m be a vector space and $\mu: V \times V \to \mathbb{R}$ a skew symmetric bilinear form. Then there is a basis $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ of V^* and $r \le m/2$ such that

$$\mu = \varepsilon_1 \wedge \varepsilon_2 + \dots + \varepsilon_{2r-1} \wedge \varepsilon_{2r}$$

<u>Proof</u>: let $\beta_1, \beta_2, \dots, \beta_m$ be any basis of V^*

write $\mu = \sum_{1 \le i < j \le m} \mu_{ij} \beta_i \wedge \beta_j$ If $\mu = 0$, done.

If $\mu \neq 0$, some coefficient is nonzero, WLOG $\mu_{12} \neq 0$

let $\varepsilon_1 = \beta_1 - \frac{1}{\mu_{12}} \sum_{i=3}^{m} \mu_{2i} \beta_i$

$\varepsilon_2 = \sum_{i=2}^{m} \mu_{1i} \beta_i$

consider $\mu' = \mu - \varepsilon_1 \wedge \varepsilon_2$, we claim μ' does not involve β_1 or β_2.

$$\varepsilon_1 \wedge \varepsilon_2 = \left[\beta_1 - \frac{1}{\mu_{12}}\left(\mu_{23}\beta_3 + \mu_{24}\beta_4 + \cdots \right)\right] \wedge \left[\mu_{12}\beta_2 + \mu_{13}\beta_3 + \cdots\right]$$

$$= \mu_{12}\beta_1\wedge\beta_2 + \mu_{13}\beta_1\wedge\beta_3 + \cdots + \mu_{1m}\beta_1\wedge\beta_m$$
$$- \left(\mu_{23}\beta_3 + \mu_{24}\beta_4 + \cdots + \mu_{2m}\beta_m\right)\wedge\beta_2 + (\text{terms w/o } \beta_1 \text{ or } \beta_2)$$
$$= \underbrace{\sum_{1<j} \mu_{1j}\beta_1\wedge\beta_j + \sum_{2<j} \mu_{2j}\beta_2\wedge\beta_j}_{\text{These are all terms that involve } \beta_1 \text{ or } \beta_2} + (\text{terms w/o } \beta_1 \text{ or } \beta_2)$$

Thus $\mu' = \mu - \varepsilon_1\wedge\varepsilon_2$ does not involve β_1 or β_2

Observe that $\varepsilon_1, \varepsilon_2, \beta_3, \beta_4, \ldots, \beta_m$ are linearly independent.

If $\mu' = 0$, done, otherwise repeat w/ μ' inplace of μ.
By induction, The lemma is proved.

To deduce the theorem, note that $\omega = \varepsilon_1\wedge\varepsilon_2 + \cdots + \varepsilon_{2r-1}\wedge\varepsilon_{2r}$ is degenerate unless $2r = m$. This implies that a symplectic vector space is even dimensional. The desired basis is given by setting $\alpha_i = \varepsilon_{2i-1}$, $\beta_i = \varepsilon_{2i}$ and letting a_i, b_i be the dual basis of V. $\blacksquare$

<u>Subspaces</u>: Let (V, ω) be a symplectic vector space.
If $W \subset V$ is a subspace, the ω-orthogonal space is $W^\perp$

$$W^\perp := \{ v \in V \mid \forall w \in W,\ \omega(v, w) = 0 \}$$

W is $\begin{cases} \text{isotropic if} & W \subseteq W^\perp \\ \text{coisotropic if} & W^\perp \subseteq W \\ \text{Lagrangian if} & W = W^\perp \\ \text{symplectic if} & W \cap W^\perp = \{0\} \end{cases}$

<u>Exercise</u>: $\dim W = 1 \Rightarrow W$ is isotropic
$\operatorname{codim} W = 1 \Rightarrow W$ is coisotropic

Recall map $\tilde{\omega} : V \to V^*$, which is an isomorphism.
There is a restriction $\text{res}: V^* \to W^*$
Consider $\text{res} \circ \tilde{\omega} : V \to W^*$
kernel $= W^\perp$, image $=$ everything b/c $\tilde{\omega}$ is isomorphism.
Therefore $\dim W^\perp + \dim W^* = \dim V$
$\dim W^\perp + \dim W = \dim V$
So W and $W^\perp$ have complementary dimension in V.

It is obvious that $W \subseteq (W^\perp)^\perp$.
By counting dimension, $W = (W^\perp)^\perp$.

W isotropic $W \subseteq W^\perp \Rightarrow \dim W \leq \dim W^\perp \Rightarrow 2\dim W \leq \dim W + \dim W^\perp = \dim V$
so $\dim W \leq \dim V/2$
W coisotropic $\Rightarrow \dim W \geq \dim V/2$
W Lagrangian $=$ (isotropic & coisotropic) $\Rightarrow \dim W = \dim V/2$

<u>Homework</u>: Lagrangian subspaces are the maximal isotropic subspaces. That is, they are the maximal elements in the poset of all isotropic subspaces partially ordered by inclusion.

Examples: $\mathbb{R}^{2n}$ basis $a_1, b_1, \dots, a_n, b_n$ $\omega = \alpha_1 \wedge \beta_1 + \cdots + \alpha_n \wedge \beta_n$

isotropic: $\mathrm{Span}\{a_1, a_2, \dots, a_r\}$, $\mathrm{Span}\{a_1, b_2\}$

Coisotropic: $\mathrm{Span}\{a_1, a_2, \dots, a_n, b_1\}$

Lagrangian $\mathrm{Span}\{a_1, \dots, a_n\}$

Symplectic $\mathrm{Span}\{a_1, b_1\}$

Prop Suppose $\{a_1, \dots, a_n\}$ span a Lagrangian subspace L of V. Then this basis may be extended to a symplectic basis of V.

Proof: Homework.

Hint: consider $\tilde{\omega}|_L : L \hookrightarrow V^*$. The image lies in $(V/L)^*$, and $\tilde{\omega}|_L : L \to (V/L)^*$ is an isomorphism.

Complex structures and metrics:

Let (V, ω) be a symplectic vector space.
Let $\{a_1, b_1, a_2, b_2, \dots, a_n, b_n\}$ be a symplectic basis.

Define $J : V \to V$ by $J(a_i) = b_i$, $J(b_i) = -a_i$

Define $g : V \times V \to \mathbb{R}$ by $g(a_i, a_j) = \delta_{ij} = g(b_i, b_j)$
$g(a_i, b_j) = 0$.

Both J and g actually depend on choice of basis.

Prop's
- $J^2 = -\mathrm{Id}_V$ (J is complex struct)
- $g(Ju, Jv) = g(u,v)$ (J is isometry of g)
- $\omega(u,v) = g(Ju, v)$
- $\omega(Ju, Jv) = \omega(u,v)$ (J is symplectic iso)

Def a vector space V with g, J, ω satisfying all these properties is a linear Kähler space.

J makes V into a $\mathbb{C}$ vector space: $(x + \sqrt{-1}\, y)\cdot v = xv + yJv$

Homework: Consider $h: V \times V \to \mathbb{C}$ $\quad h = g - \sqrt{-1}\,\omega$
Then h is a Hermitian inner product on V (regarded as a $\mathbb{C}$-v.s.)

Homework: let $\omega\perp$ denote ω-orthogonal space
let $g\perp$ denote g-orthogonal space
Show for $W \subseteq V$:
(i) $W^{\omega\perp} = (JW)^{g\perp}$
(ii) $W^{g\perp} = (JW)^{\omega\perp}$
(iii) $JW \cap W^{\omega\perp} = 0$

Corollary of Homework: If $L \subseteq V$ Lagrangian then $JL = L^{g\perp}$

$$(JL)^{g\perp} \underset{\text{Homework}}{=} L^{\omega\perp} \underset{\text{Lag.}}{=} L.$$ Take $g\perp$ on both sides.

Homework: use this to give another solution of problem of extending a Lagrangian basis to a symplectic basis.

First look at <u>The Lagrangian Grassmannian</u>.

$$\Lambda(V,\omega) = \{ L \subset V \mid L \text{ is Lagrangian} \}$$

The Fact is that $\Lambda(V,\omega)$ is a compact manifold of dimension $n(n+1)/2$, as we shall see.

Since (V,ω) admits g, J and $h = g - \sqrt{-1}\,\omega$, we might as well use them to study $\Lambda(V,\omega)$, even though they aren't canonically associated to ω.

<u>Homework</u>: What is $\Lambda(\mathbb{R}^2, dx\wedge dy)$?

<u>Proposition</u>: $\Lambda(V,\omega)$ is diffeomorphic to $U(n)/O(n)$

<u>Lemma</u> $\{e_1, e_2, \dots, e_n\}$ a g-orthonormal basis of a Lagrangian subspace $\Longleftrightarrow$ $\{e_1,\dots,e_n\}$ is a unitary basis of $\mathbb{R}^{2n}$ (regarded as a complex Euclidean space via complex structure J and hermitian inner product $h = g - \sqrt{-1}\,\omega$)

<u>Proof of lemma</u>: e_i unitary basis $\Leftrightarrow h(e_i,e_j) = \delta_{ij}$
$\Longleftrightarrow g(e_i,e_j) = \delta_{ij}$ and $\omega(e_i,e_j) = 0$
$\Longleftrightarrow$ e_i are g-orthonormal and span e_i is isotropic, n-dimensional hence Lagrangian. ∎

<u>Proof of prop</u>: Pick a Lagrangian L and g-orthonormal basis $\{e_1,\dots,e_n\}$ of L. For any other Lagrangian L' with basis $e_1',\dots,e_n'$, there is a unitary transformation $A \in U(V,h) \simeq U(n)$ such that $e_i' = Ae_i$.

This follows from the lemma since $\{e_i\}$ and $\{e_i'\}$ are unitary bases of (V,h).
Thus $U(n) \simeq U(V,h)$ acts transitively on $\Lambda(V,\omega)$.
The stabilizer of L is the set of unitary transformations that do not change the $\mathbb{R}$-span of $\{e_1,\dots,e_n\}$.
These are elements of $U(n)$ that have real entries when represented in the $\mathbb{C}$-basis $\{e_1,\dots,e_n\}$ of V.
A unitary matrix satisfies $\bar{A}^T A = I$. If in addition A is real, $\bar{A}=A$ so $A^T A = I$ and A is orthogonal.

$$\text{Thus, } \Lambda(V,\omega) = U(V,h)\cdot L = U(V,h)/O(L,g) = U(n)/O(n).$$

$\blacksquare$

<u>Cor</u> $\Lambda(V,\omega)$ has the structure of a smooth manifold of dimension $n(n+1)/2$.

<u>Proof</u> $\dim U(n) = n^2$ $\qquad \dim O(n) = n(n-1)/2$

<u>Prop</u> The tangent space to $\Lambda(V,\omega)$ at L is the space of symmetric bilinear forms on L

<u>Proof</u>

consider a path of subspaces $\Phi : \mathbb{R}\times L \to V$ of the form
$$\Phi_\varepsilon(x) = x + \varepsilon\phi(x) \quad \text{for some } \phi : L \to V$$

These are Lagrangian iff $\omega(x_1+\varepsilon\phi(x_1),\, x_2+\varepsilon\phi(x_2)) = 0$
$$\varepsilon\left[\omega(x_1,\phi(x_2)) + \omega(\phi(x_1),x_2)\right] + \varepsilon^2\omega(\phi(x_1)\,\phi(x_2)) = 0.$$
Thus, Φ_ε is Lagrangian to first order in ε iff:
$$\omega(x_1,\phi(x_2)) - \omega(x_2,\phi(x_1)) = 0 \quad \forall\, x_1,x_2 \in L$$
i.e., $\omega(x,\phi(y))$ is a symmetric bilinear form on L

Consider the determinant map $\det : U(n) \to S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$
det maps $O(n)$ to $\{\pm 1\}$, so det does ~~not~~ define a map on $\Lambda(n) = U(n)/O(n)$. However: $\det^2$ does

$$\det^2 : U(n)/O(n) \longrightarrow S^1$$

Homework: The fiber $(\det^2)^{-1}(1)$ is $SU(n)/SO(n) \subset U(n)/O(n)$

Prop $\det^2$ induces an isomorphism $\pi_1(U(n)/O(n)) \to \pi_1(S^1) \simeq \mathbb{Z}$

Facts: $SU(n)$ is simply connected, and $SO(n)$ is connected.

Lemma: $\pi_1(SU(n)/SO(n))$ is trivial.
Proof: Exact sequence of a fibration

$$\{*\} = \pi_1(SU(n)) \to \pi_1(SU(n)/SO(n)) \longrightarrow \pi_0(SO(n)) = \{*\}$$

Proof of prop: Exact sequence of fibration $SU(n)/SO(n) \to U(n)/O(n) \xrightarrow{\det^2} S^1$

$$\{*\} = \pi_1(SU(n)/SO(n)) \longrightarrow \pi_1(U(n)/O(n)) \xrightarrow{\det^2} \pi_1(S^1) \to \pi_0(SU(n)/SO(n)) = \{*\}$$

Cor: $(\det^2)^* : H^1(S^1; \mathbb{Z}) \longrightarrow H^1(\Lambda(n); \mathbb{Z})$ is an isomorphism
Definition: $\mu := (\det^2)^*[d\theta] \in H^1(\Lambda(n); \mathbb{Z})$ is called the universal Maslov class.

We could continue studying this Lagrangian Grassmannian, but let's move on to recalling some facts about symplectic manifolds.