Lagrangian Floer homology: First lecture

M smooth manifold, $\omega \in \Omega^2(M)$ symplectic form

(mean ω is nondegenerate skew-symmetric $\mathbb{R}$-bilinear form on tangent spaces, and satisfies $d\omega = 0$)

e.g. (1) $\mathbb{R}^{2n} = (x_1, y_1, x_2, y_2, \dots)$, $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$

(2) cotangent bundle T^*Q, ω_{can}

(3) Kähler manifold (X, g) $\quad \omega(X,Y) = g(iX, Y)$
then $d\omega = 0$ expresses the Kähler condition
($\Rightarrow$ Euclidean to 2^{nd} order in holomorphic coordinates) (i denote cx structure tensor)

Class (3) contains all quasi-projective varieties.
(Because $\mathbb{CP}^n$ is Kähler)

<u>Natural conditions on submanifolds</u>: $S \subset M$
S is <u>isotropic</u> if $\omega|_S = 0$

S is <u>lagrangian</u> if it is isotropic and half-dimensional
(the maximal dimension allowed by linear algebra)

<u>Hamiltonian vector fields</u>: Given $H \in C^\infty(M, \mathbb{R}) \rightsquigarrow dH \in \Omega^1(M)$
Define a vector field $X_H \in \mathfrak{X}(M)$ by the relation

$$\omega(Y, X_H) = dH(Y)$$

A flow on M is called Hamiltonian if it is defined by a time-dependent Hamiltonian function $H : \mathbb{R} \times M \to \mathbb{R}$

Let H_t denote restriction to $\{t\} \times M$

X_t = Ham. V.F. associated to H_t

Define the flow $\varphi_t(x)$ by $\frac{d}{dt}(\varphi_t(x)) = X_t(\varphi_t(x))$

Then $\varphi_1 : M \to M$ is a <u>Hamiltonian</u> <u>Diffeomorphism</u>.

<u>Prop:</u> Hamiltonian diffeomorphisms preserve ω: $\varphi_1^* \omega = \omega$

<u>Thm</u> Hamiltonian diffeomorphisms form a <u>subgroup</u>

$$\mathrm{Ham}(M,\omega) \subset \underbrace{\mathrm{Symp}(M,\omega)}_{= \{\varphi \in \mathrm{Diff}(M) \mid \varphi^*\omega = \omega\}} \subset \mathrm{Diff}(M)$$

- Lagrangian Floer Homology is an intersection theory for Lagrangian submanifolds $L \subset M$.
- Ordinary intersection theory measures properties of the intersection that are unchanged by continuous deformation

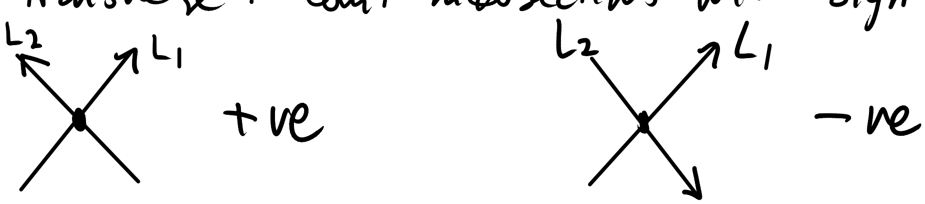
M closed oriented of dim $2n$

$L_1, L_2 \subset M$ half dimensional closed oriented of dim n

intersection # $I(L_1, L_2)$

Assuming transverse: count intersections with sign

$n=1$:



$$\begin{tikzcd}
{[L_1]\otimes[L_2]} \arrow[rrr, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & & I(L_1,L_2)\in\mathbb{Z} \arrow[d, phantom, "\parallel"] \\
H_n(M;\mathbb{Z})^{\otimes 2} \arrow[r, "PD"] & H^n(M;\mathbb{Z})^{\otimes 2} \arrow[r, "\cup"] & H^{2n}(M;\mathbb{Z}) \arrow[r, "PD"] & H_0(M;\mathbb{Z})
\end{tikzcd}$$

- Lagrangian intersection theory is concerned with properties of the intersection that are constant under Hamiltonian deformations. these are those properties that are "symplectically essential."

The invariant is not a <u>number</u> $I(L_1,L_2)$ but a <u>vector space</u>

$$HF^*(L_1,L_2)$$

(grading is a tricky issue, assume $\mathbb{Z}_2$ for now)

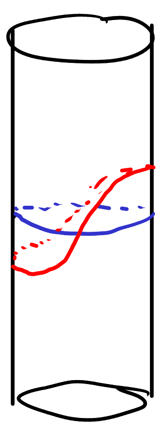
This move is an example of the "CATEGORIFICATION" philosophy

We recover $I(L_1,L_2)$ by taking Euler characteristic (super dimension)

$$I(L_1,L_2) = \dim HF^{even}(L_1,L_2) - \dim HF^{odd}(L_1,L_2)$$
$$= \operatorname{sdim} HF^*(L_1,L_2)$$

$HF^*(L_1,L_2)$ may be nontrivial even though $I(L_1,L_2)=0$

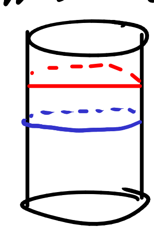
T^*S^1



L_1 = zero-section

L_2 = Hamiltonian push-off of zero section

Clearly, the intersections can be destroyed by a continuous deformation of L_2



However, $HF^*(L_1, L_2)$ has rank 2, $\left.\begin{matrix} HF^0(L_1,L_2) \\ HF^1(L_1,L_2) \end{matrix}\right\}$ both rank 1

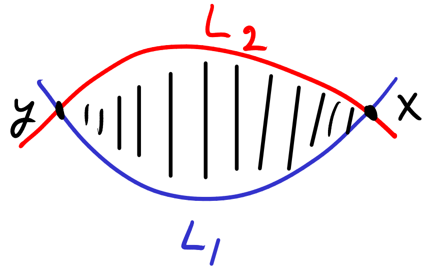
This implies that L_2 cannot be disjoined from L_1 by any Hamiltonian diffeomorphism.

How is $HF^*(L_1, L_2)$ defined? It is the homology of a complex $CF^*(L_1, L_2)$.

Roughly:

$CF^*(L_1, L_2)$ is spanned by intersection points

Floer differential or boundary map counts pseudoholomorphic curves $u: \mathbb{R} \times [0,1] \to M$ satisfying boundary and asymptotic conditions:



It is very technically demanding to make sense of this.

Where did the idea come from? Morse theory for the symplectic action functional on the space of paths

$$\Omega(L_1, L_2) = \{ \gamma : [0,1] \to M \mid \gamma(0) \in L_1, \gamma(1) \in L_2 \}$$

$$\mathcal{A} : \Omega(L_1, L_2) \to \mathbb{R} \text{ s.t. } d\mathcal{A}_\gamma(X) = \int_0^1 \omega(X, \dot{\gamma})\, dt$$

What is this used for? Symplectic topology:

- study displaceability of Lagrangians as in example
- Can also translate fixed points of symplectic diffeomorphisms into Lagrangian intersections.

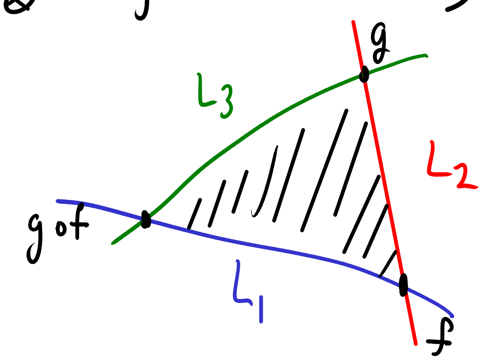
- Can probe topology of symplectic diffeomorphism group by looking at how Lagrangians move around.

- Symplectic diffeomorphisms come up as monodromy of algebraic families, leading to a "Symplectic Picard-Lefschetz theory".

- Invariants in low dimensional topology often have symplectic interpretations: Seiberg-Witten as Heegaard Floer, Instanton homology and Khovanov homology. We recover invariant as Lagrangian Floer homology in an associated space.

- HF^* supports product structures, the first of which is

$$HF^*(L_2, L_3) \otimes HF^*(L_1, L_2) \longrightarrow HF^*(L_1, L_3)$$
$$g \otimes f \longrightarrow g \circ f$$



(various details omitted)

This gives us a category where the objects are L_i's, the set of morphisms $L_i \to L_j$ is $HF^*(L_i, L_j)$ and the composition of morphisms is this product. This is called the Donaldson category. By considering "higher products", we obtain the Fukaya A_∞-category. It plays a fundamental role in homological mirror symmetry.

Homework: $M = S^2 = \{x^2+y^2+z^2=1\} \subset \mathbb{R}^3$

ω = area form induced from round metric,

$L = S^2 \cap \{z=0\}$ (the "equator")

Observe that M is symplectic and L is Lagrangian.

(1) Find $\varphi \in \mathrm{Diff}(M)$ such that $\varphi(L) \cap L = \emptyset$

(2) Prove that for any $\varphi \in \mathrm{Symp}(M)$, $\varphi(L) \cap L \neq \emptyset$.

(3) Investigate same questions for $L_h = S^2 \cap \{z=h\}$, where $0<h<1$.