Salamon: lectures on Floer homology

Morse Theory

(Reference Milnor: { Morse Theory ; Lec. on h-cobordism thm })

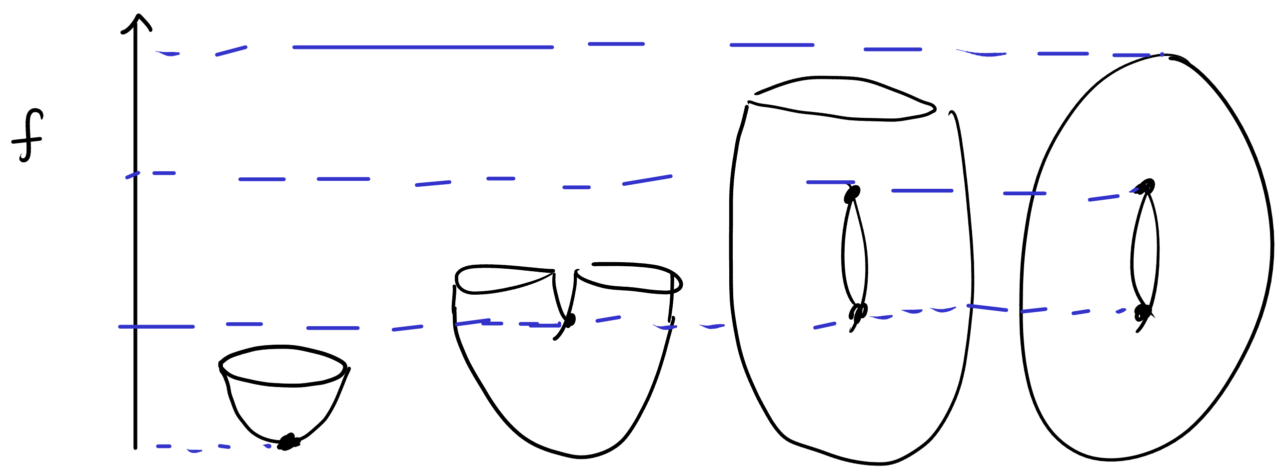
Floer: Witten's complex....

Besides the pseudo-holomorphic curves of Gromov, this is the other key precursor to Floer Homology

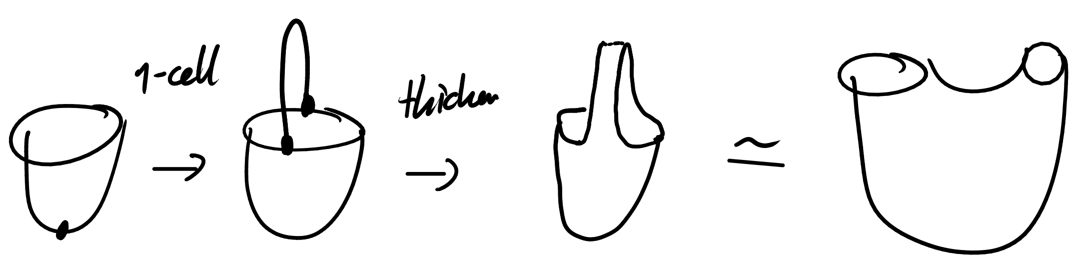
<u>Idea</u> Given a space (manifold) M and a function $f: M \to \mathbb{R}$ look at sublevel sets

$$M^a := f^{-1}(-\infty, a]$$

with an eye towards how the topology depends on a.



Homotopy type changes when a is a critical value of f
In fact it changes by a cell attachment



<u>Def</u> A <u>critical point</u> of $f: M \to \mathbb{R}$ is a point such that $df: T_pM \to T_{f(p)}\mathbb{R}$ vanishes. The value $f(p)$ is then called a <u>critical value</u>

Let's prove a precise statement

<u>Theorem</u> Let M be smooth mfld, $f: M \to \mathbb{R}$ smooth. Let $a < b$ and suppose $f^{-1}[a,b]$ is compact, and <u>contains no critical points of f</u>.
Then M^a is a deformation retraction of M^b, so inclusion $M^a \hookrightarrow M^b$ is homotopy equivalence.

<u>Proof</u> pick Riemannian metric g on M. gradient ∇f is defined by $g(\nabla f, X) = X \cdot f$ $\forall$ vector fields X

Let $\alpha: M \to \mathbb{R}$ denote $\alpha(p) = g(\nabla f_p, \nabla f_p) = \|\nabla f_p\|_g^2$

Then $\frac{1}{\alpha}$ exists on $f^{-1}[a,b]$, as this set contains no critical points.

Let Z be a vector field that equals $-\frac{1}{\alpha}\nabla f$ on $f^{-1}[a,b]$ and which vanishes outside a compact neighborhood of $f^{-1}[a,b]$ (using $f^{-1}[a,b]$ is compact here.)

Since Z is compactly supported, it is complete and its flow φ^t exists for all time.

Compute, in $f^{-1}[a,b]$, $\mathcal{L}_Z f = Z \cdot f = -\frac{1}{\alpha}\nabla f \cdot f = \frac{-1}{\alpha} g(\nabla f, \nabla f) = -1$

Thus f decreases at unit rate along trajectories of Z, so φ^t maps $f^{-1}(c)$ to $f^{-1}(c-t)$ for $a \leq c-t < c \leq b$

In particular φ^{b-a} maps M^b into M^a, and $\{\varphi^t\}_{t\in[0,b-a]}$ is desired deformation retraction. $\boxed{\checkmark}$

We already see the crucial geometric structures:

- critical points $df = 0$
- gradient flow $\dot{x} = -\nabla f$

<u>Local structure at a critical point</u>

$f : M \to \mathbb{R}$, $p \in M$ critical pt., $df_p = 0$

We work in an arbitrary local coordinate system $(x^1, \dots, x^n) \in U$ centered at $p \in M$ $\quad (p \leftrightarrow (0, \dots, 0))$

$df_p = 0$ means $\dfrac{\partial f}{\partial x^i}(0) = 0 \quad \forall i = 1, \dots, n$

The critical point is <u>nondegenerate</u> if the Hessian = matrix of 2nd derivs is nondegenerate

$$\mathrm{Hess}_p(f) = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}(0) \right)_{i,j = 1, \dots, n}$$

- This is a symmetric matrix.
- nondegenerate $\Leftrightarrow$ $\det \mathrm{Hess}_p(f) \neq 0$.

This is independent of the coordinate system, since $\mathrm{Hess}_p(f)$ has an intrinsic formulation.

Let $X_p, Y_p \in T_pM$. Extend them arbitrarily to vector fields X, Y throughout a neighborhood of p.

Define $\mathrm{Hess}_p(f)(X_p, Y_p) = \left[X \cdot (Y \cdot f) \right](p)$

- Clearly tensorial in X i.e. only depends on X_p, not the extension.

- Symmetric in X and Y because
$$X\cdot(Y\cdot f) - Y\cdot(X\cdot f) = [X,Y]\cdot f = df_p([X,Y]) = 0$$
since p critical

Thus $\mathrm{Hess}_p(f)(X_p, Y_p) = Y\cdot(X\cdot f)$, which is clearly tensorial in Y.

We conclude that $\mathrm{Hess}_p(f)$ is a well-defined symmetric bilinear form on T_pM

If $X = X^i \frac{\partial}{\partial x^i}$ $Y = Y^j \frac{\partial}{\partial x^j}$

$$X\cdot(Y\cdot f) = \sum_{i,j=1}^n X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j}$$ so in the frame $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$

$\mathrm{Hess}_p(f)$ is represented by matrix $\frac{\partial^2 f}{\partial x^i \partial x^j}$.

Spectral theory of real symmetric matrices $\Rightarrow$ $\mathrm{Hess}_p(f)$ is diagonalizable with real $\underbrace{\text{eigenvalues and eigenvectors}}_{\text{these have geometry}}$.

Nullity = # of zero eigenvalues (= 0 if nondegenerate)

Index = # of negative eigenvalues

<u>Morse lemma</u>: If p is a nondegenerate critical point of f, There is a local coordinate system $(y^1, \dots, y^n)$ such that
$$f(y^1, \dots, y^n) = f(p) - \sum_{i=1}^{\lambda} (y^i)^2 + \sum_{i=\lambda+1}^{n} (y^i)^2$$
where λ is the index of p = index of $\mathrm{Hess}_p(f)$

I.e., we can actually "diagonalize" f in a neighborhood.

Such coordinate system is called a <u>standard Morse chart.</u>

To look at gradient flow, lets choose a metric g which is also standard in a Morse chart

so: $f = -(y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2$

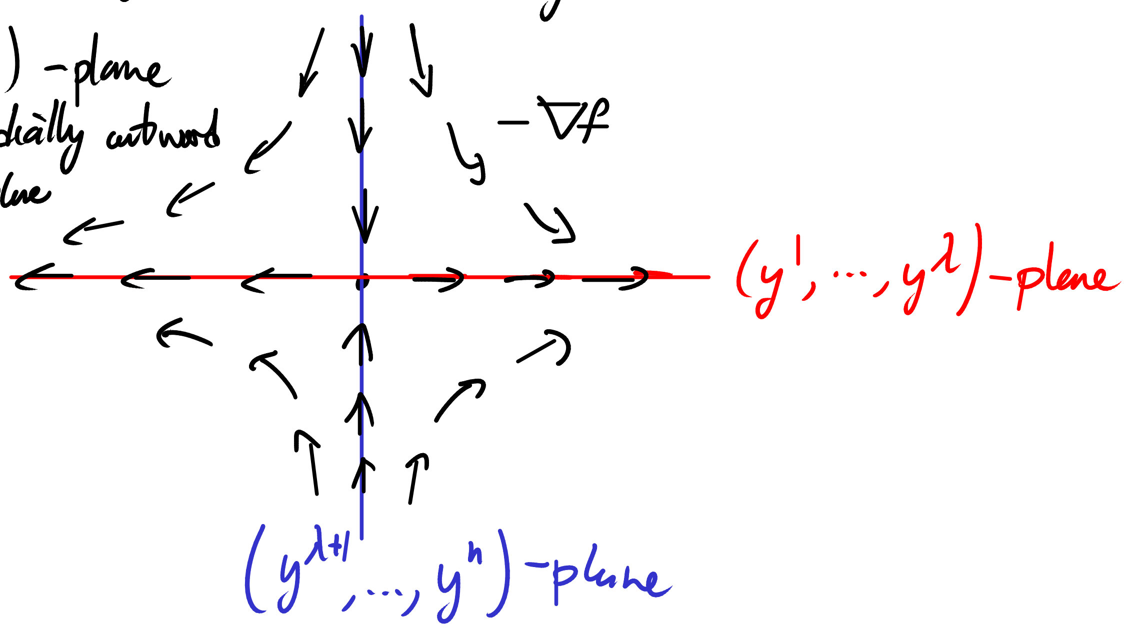
$$g = \sum_{i=1}^n (dy^i)^2$$

$$df = -2y^1 dy^1 - \cdots + 2y^{\lambda+1} dy^{\lambda+1} + \cdots$$

$$-\nabla f = +2y^1 \frac{\partial}{\partial y^1} + \cdots - 2y^{\lambda+1} \frac{\partial}{\partial y^{\lambda+1}} - \cdots$$

on the $(y^1, \ldots, y^\lambda)$-plane $-\nabla f$ points radially outward

on the $(y^{\lambda+1}, \ldots, y^n)$ plane $-\nabla f$ points radially inward



In this local model, the $(y^1, \ldots, y^\lambda)$-plane (where $y^{\lambda+1} = \cdots = y^n = 0$) is an invariant subset for the $-\nabla f$ flow. It is called the unstable manifold / descending (stable) manifold / variété stable sortant

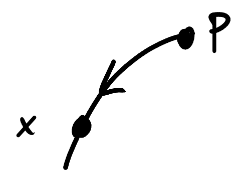
The $(y^{\lambda+1}, \ldots, y^n)$-plane is called stable manifold / ascending (stable) manifold / variété stable entrant.

Confusingly, the unstable manifold is <u>attracting</u> for nearby trajectories.

More generally, we may define the unstable manifold for p to be the union of all trajectories of $-\nabla f$ that "emanate" from p

let φ^t denote flow of $-\nabla f$

$$W^u(p) = \left\{ x \in M \;\middle|\; \lim_{t \to -\infty} \varphi^t(x) = p \right\}$$



(Because $-\nabla f = 0$ at p, it will take infinite time for x to reach p.)

Conversely the stable manifold is

$$W^s(p) = \left\{ x \in M \;\middle|\; \lim_{t \to \infty} \varphi^t(x) = p \right\}$$

<u>Theorem</u> (1) $W^u(p)$ and $W^s(p)$ are diffeomorphic to balls of dimensions $\lambda = \text{index}(p)$ and $n - \lambda$ respectively.

(2) let E_p^+ and E_p^- denote the positive and negative eigenspaces of $\text{Hess}_p(f)$ at p (w.r.t. metric g)

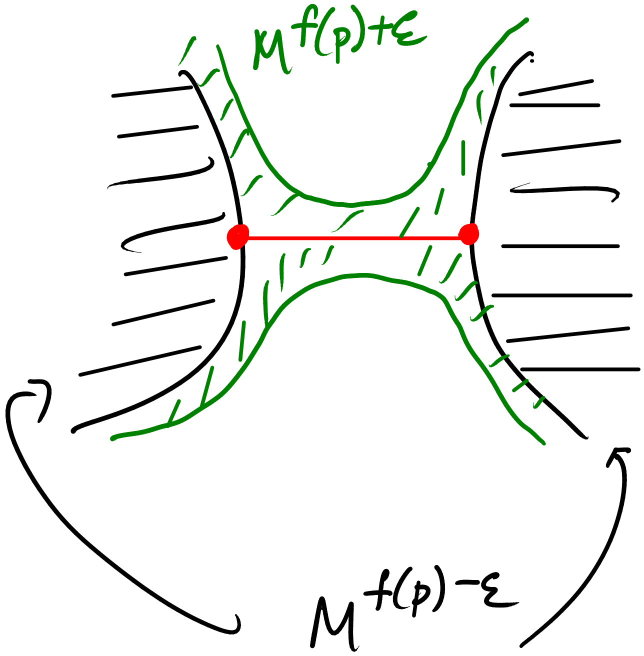
Then there are natural isomorphisms

$$T_p W^u(p) \simeq E_p^- \qquad T_p W^s(p) \simeq E_p^+$$

<u>Proof</u> if the metric is standard in some Morse chart, these statements are clear as W^u actually coincides with E_p^- in a neighborhood, and similarly for W^s and E_p^+.

The metric may be taken standard at the level of T_pM because the eigenvectors of $\mathrm{Hess}_p(f)$ are orthogonal. Then the linearization of the gradient flow at p is as before. $\square$

Key insight: Consider $M^a = f^{-1}(-\infty, a]$
as we pass the critical level $a = f(p)$
the cell which gets attached is precisely $W^u(p)$



What happens to homology : Pair $(M^{a+\varepsilon}, M^{a-\varepsilon})$

$$H_*(M^{a+\varepsilon}, M^{a-\varepsilon}; k) = \begin{cases} k & * = \lambda \\ 0 & \text{o/w} \end{cases}$$

Sequence of pair: only interesting part is

$$0 \to H_\lambda(M^{a-\varepsilon}) \to H_\lambda(M^{a+\varepsilon}) \to H_\lambda(M^{a+\varepsilon}, M^{a-\varepsilon}) \xrightarrow{\partial} H_{\lambda-1}(M^{a-\varepsilon}) \to H_{\lambda-1}(M^{a+\varepsilon}) \to 0$$

need to determine this map.

To compute ∂, we need to express class of the sphere

$$W^u(p) \cap f^{-1}(a-\varepsilon) \simeq S^{\lambda-1} \quad \text{in} \quad H_{\lambda-1}(M^{a-\varepsilon})$$

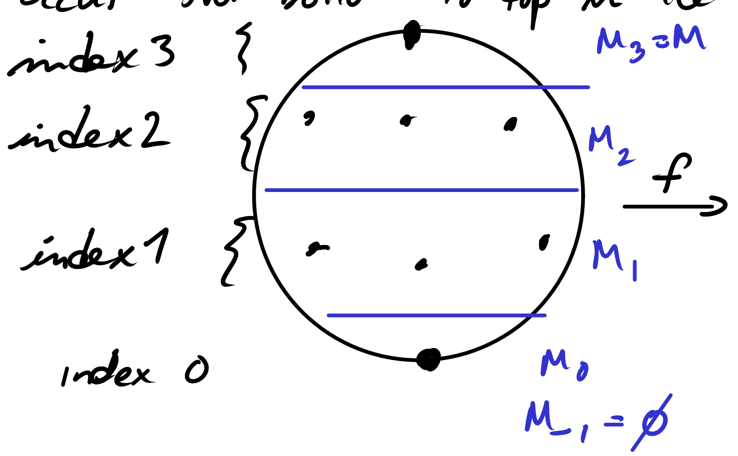
in terms of some basis, say.

We can generalize the question to compute the entire homology directly from the Morse theory.

Self-indexing Morse functions

Strictly: $\forall p \in \mathrm{Crit}(f) \qquad f(p) = \mathrm{index}(p)$

But all that really matters is that the critical points occur from bottom to top in the order of their index e.g.



Let $n_i = \#$ crit pts of index i

Choose M_i = some sublevel set containing the critical points of index $\leq i$

Homologically speaking, the filtration M_i has the same property as the filtration of a CW complex by skeleta.

$$H_*(M_i, M_{i-1}) = \begin{cases} \mathbb{Z}^{n_i} & * = i \\ 0 & \text{o/w} \end{cases}$$

This means we can compute the homology just as we would for the CW case ("cellular homology")

Triple (M_i, M_{i-1}, M_{i-2})

$$0 \to C_*(M_{i-1})/C_*(M_{i-2}) \to C_*(M_i)/C_*(M_{i-2}) \to C_*(M_i)/C_*(M_{i-1}) \to 0$$

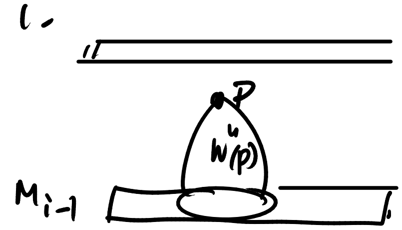
Connecting homomorphism. $\partial: H_i(M_i, M_{i-1}) \to H_{i-1}(M_{i-1}, M_{i-2})$

Define $C_i^{morse} = H_i(M_i, M_{i-1})$

$\partial: C_i^{morse} \to C_{i-1}^{morse}$ = connecting homomorphism for (M_i, M_{i-1}, M_{i-2})

Note: rank $C_i^{morse} = n_i$ = # crit pts of index i.

The generators are the classes of $W^u(p)$



$$\begin{array}{c} H_i(M_i, M_{i-1}) \\ \rotatebox{90}{$\in$} \\ [W^u(p)] \end{array} \to [W^u(p) \cap \partial M_{i-1}]$$

<u>Theorem</u> (Lemma 7.2 of Milner h-cobord.)

For Z a $(i-1)$ cycle in ∂M_{i-1}, the class in $H_{i-1}(M_{i-1}, M_{i-2})$ satisfies

$$[Z] = \sum_{\substack{p \text{ crit} \\ \text{of index} \\ i-1}} (Z \cdot W^s(p)) [W^u(p)]$$

Actually, for this to make sense, we need to orient $W^u(p)$ and $W^s(p)$. It suffices to orient E_p^+ and E_p^- the positive and negative eigenspaces at each critical point p.

By convention, we require $W^u(p) \cdot W^s(p) = +1$

Proof that this computes $H_*(M)$.

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
H_{n+1}(M_{n+1}, M_n) \arrow[r] \arrow[dr, "\partial"'] & H_n(M_n, M_{n-2}) \arrow[r] \arrow[d] & H_n(M_{n+1}, M_{n-2}) \arrow[r] & 0 \\
 & C_n = H_n(M_n, M_{n-1}) \arrow[d, "\partial"'] & \\
 & C_{n-1} = H_{n-1}(M_{n-1}, M_{n-2}) &
\end{tikzcd}$$

$H_{n+1}(M_{n+1}, M_n) = C_{n+1}$

$Z_n \simeq H_n(M_n, M_{n-2})$

$Z_n / B_n \simeq H_n(M_{n+1}, M_{n-2})$

<u>Claim</u> $H_n(M_{n+1}, M_{n-2}) \simeq H_n(M)$

- Since $H_*(M_n, M_{n-1})$ is concentrated in degree i

(i) $H_i(M_n)$ vanishes for $i > n$

(ii) for $i < n$, $H_i(M_n) \simeq H_i(M_{n+1})$

This is true using exact sequences of pairs and induction

Then $$H_n(M_{n-2}) \to H_n(M_{n+1}) \to H_n(M_{n+1}, M_{n-2}) \to H_{n-1}(M_{n-2})$$

$$\underset{0}{\shortparallel} \longleftarrow \text{by (i)} \longrightarrow \underset{0}{\shortparallel}$$

$$\therefore \quad H_n(M_{n+1}) \simeq H_n(M_{n+1}, M_{n-2}) = Z_n / B_n$$

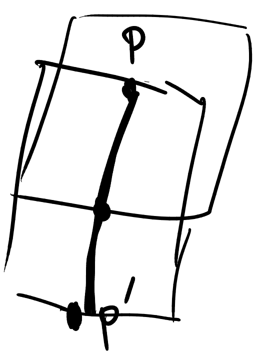
Also $H_n(M_{n+1}) \simeq H_n(M_{n+2}) \cdots\cdots \simeq H_n(M)$ by (ii) $\boxtimes$

Now we make a trivial observation:

The differential $\partial : C_n^{morse} \to C_{n-1}^{morse}$ is computed thus:

- Take p a critical pt of index n
- Look at unstable manifold $W^u(p)$
- Intersect this with ∂M_{n-1} to get a $(n-1)$-sphere $S^u(p)$ in M_{n-1}
- Intersect $S^u(p)$ with $W^s(p')$ for p' of index $(n-1)$

suppose this intersection is transverse! (Morse-Smale condition)



A point $z \in S^u(p) \cap W^s(p')$ lies on a trajectory $\gamma(t)$ of $-\nabla f$ such that

$\lim_{t \to -\infty} \gamma(t) = p$ since $z \in S^u(p) \subset W^u(p)$

$\lim_{t \to \infty} \gamma(t) = p'$ since $z \in W^s(p')$

Thus z corresponds precisely to a flow line / trajectory joining p to p'

When the intersection is transverse, $\partial : C_n^{morse} \to C_{n-1}^{morse}$ is <u>counting</u> such trajectories! (with sign)

This interpretation is dubbed "Witten's complex" by Floer (but it is essentially cellular homology of the morse stratification, as we have seen)

The conditions that intersections of stable / unstable manifolds be transverse is called the Morse-Smale condition, and it is a generic condition on pairs (f, g)

$$\underset{\text{function}}{f}, \quad \underset{\text{metric}}{g}$$

So can redefine Morse complex for a morse-smale pair (f,g)

$$C_n^{morse} = \bigoplus_{\substack{p \text{ critical} \\ \text{point of} \\ \text{index } n}} \mathbb{Z}\langle p \rangle$$

$$\partial : C_n^{morse} \longrightarrow C_{n-1}^{morse} \qquad \partial p = \sum_q n(p,q)\, q$$

$n(p,q) =$ (signed) count of trajectories connecting $p \to q$

$$\left\{ \gamma : \mathbb{R} \to M \;\middle|\; \begin{array}{l} \dot\gamma = -\nabla f \\ \gamma(-\infty) = p \quad \gamma(\infty) = q \end{array} \right\}$$

As a consequence of what has been said so far,

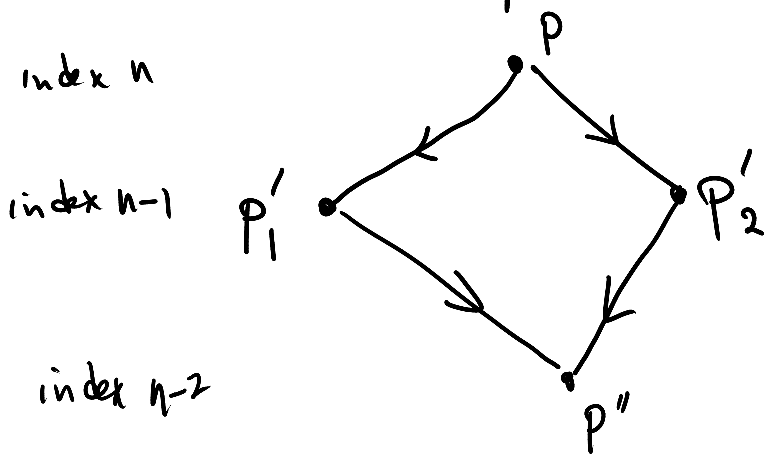
$$\begin{tikzcd} C_n^{morse} \arrow[r] \arrow[rr, bend right, "\partial^2 = 0"'] & C_{n-1}^{morse} \arrow[r] & C_{n-2}^{morse} \end{tikzcd}$$

What is the geometric reason for this?

$$\partial^2 p = \sum_{p', p''} n(p,p')\, n(p',p'')\, p''$$

Need to see: $\sum_{p'} n(p,p')\, n(p',p'') = 0$

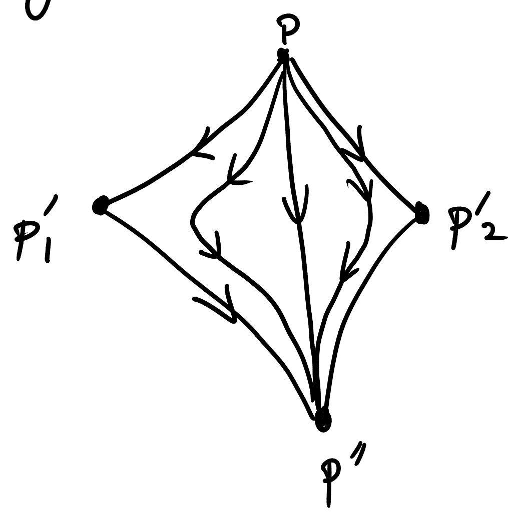
$\sum_{p'} n(p,p')\, n(p',p'')$ represents "broken flow lines"



Idea: look at all flow lines from p to p''.
By morse-smale condition these fill out a surface
(since index difference = 2)

Now: take the closure of $W^u(p) \cap W^s(p'')$

The boundary consists of broken flow lines, stopping at an intermediate critical point p'.



The trajectories $p \to p''$ "fill in" the square.

We must also show that every broken trajectory occurs as a boundary in this way (requires gluing trajectories)

Now we conclude that "pairs of trajectories come in pairs"

This shows that $\partial^2 = 0$ modulo 2. If we get orientations into the story, we will also get cancelling signs.

(Recall that orientation data are orientations of all stable and unstable manifolds)

Slight change in language: Space of gradient flow lines has a $\mathbb{R}$-reparametrization symmetry

$c : \gamma(t) \mapsto \gamma(t+c)$, which is a free action.

The "moduli space" of flow lines is the quotient by this action

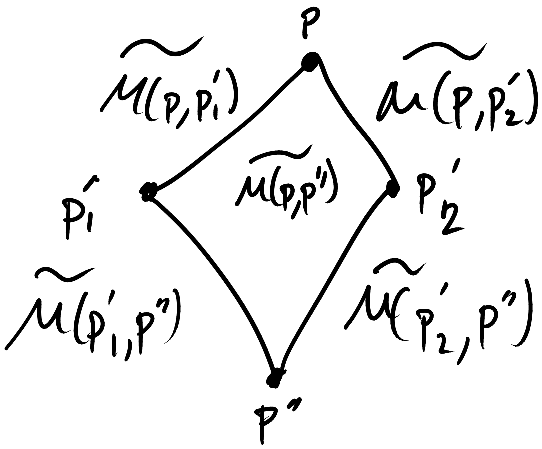
$$\mathcal{M}(x,y) = (\text{flowlines } x \to y)/\mathbb{R}$$

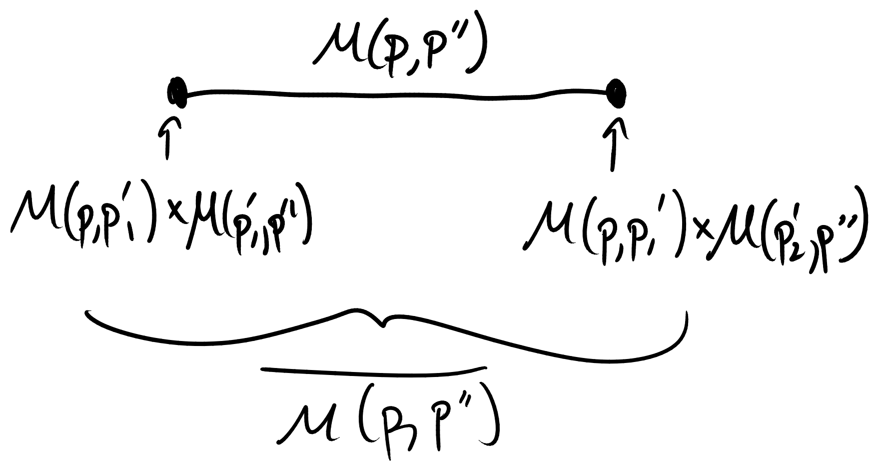
For Morse-Smale flow:

$$\dim \mathcal{M}(x,y) = \mathrm{index}(x) - \mathrm{index}(y) - 1$$

So $\partial : C_n^{morse} \to C_{n-1}^{morse}$ counts points in the zero dimensional moduli spaces.

To prove $\partial^2 = 0$, we used compactifications of the 1-dimensional moduli spaces.





Very schematically: $\partial \mathcal{M} = \mathcal{M} \times \mathcal{M}$