(Cf. Schwarz, Morse Homology)

Compactification of Morse flow lines

Recall M compact, $f: M \to \mathbb{R}$ Morse
g a "generic" metric, namely one which achieves the <u>Morse-Smale condition</u>:

(MS) $W^u(x) \pitchfork W^s(y)$ for each pair $x, y \in Crit(f)$

Since each set $W^u(x)$ and $W^s(y)$ is a smooth manifold without boundary (noncompact of course), their transverse intersection

$$\mathcal{M}(x,y) = W^u(x) \cap W^s(y) \text{ is also}$$

a smooth manifold without boundary.

$$\mathcal{M}(x,y) = \text{"parametrized flow lines } x \longrightarrow y\text{"}$$

(In Floer theory, even the construction of $\mathcal{M}(x,y)$ requires some analysis. Right now we are relying on the existence theory for the ODE $\dot{u} = -\nabla f \circ u$ as well as the claimed genericity of the Morse-Smale condition)

In order to do geometry with $\mathcal{M}(x,y)$, we need to find a natural compactification.

The space $\mathcal{M}(x,y)$ has a feature that is in tension with compactness, namely, a free action of a noncompact group!

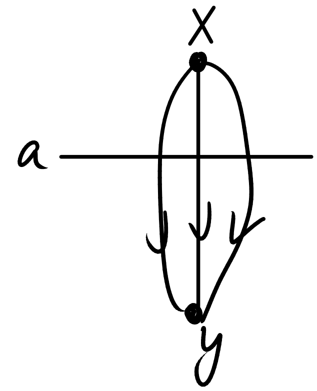
$$G = (\mathbb{R}, +) \text{ by reparametrization.}$$

$\gamma \in \mathcal{M}(x,y)$ is a map $\gamma : \mathbb{R} \to M$ s.t. $\dot{\gamma} = -\nabla f \circ \gamma$

Now let $\tau \in \mathbb{R}$ act by

$$(\gamma \bullet \tau)(t) = \gamma(t+\tau)$$

- This maps solutions of $\dot{\gamma} = -\nabla f \circ \gamma$ to solutions.
- As long as $x \neq y$, $\mathcal{M}(x,y)$ consists of nonconstant trajectories, and the $\mathbb{R}$-action on $\mathcal{M}(x,y)$ is free.
- This action admits smooth slices:
  Let a be a regular value of f
  $$f(y) < a < f(x)$$



There is an evaluation map $\varphi : \mathcal{M}(x,y) \to \mathbb{R}$

$$\varphi(\gamma) = f(\gamma(0))$$

Then $\mathcal{M}(x,y)^a = \varphi^{-1}(a) \subset \mathcal{M}(x,y)$
is a smooth submanifold.

<u>Lemma</u> The map $\Psi^a : \mathbb{R} \times \mathcal{M}(x,y)^a \to \mathcal{M}(x,y)$
$$\Psi^a(\tau, \gamma) = \gamma \bullet \tau$$
is an $\mathbb{R}$-equivariant diffeomorphism.

Using such a slice, the quotient obtains the structure of a smooth manifold via

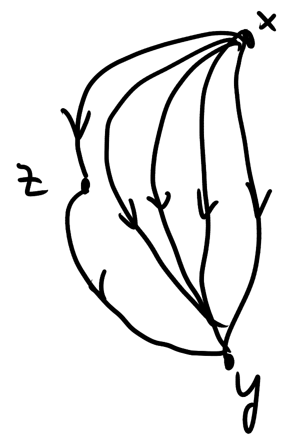
$$\overline{\mathcal{M}}(x,y) := \mathcal{M}(x,y)/\mathbb{R} \approx \mathcal{M}(x,y)^a$$

for any value $a \in (f(y), f(x))$

The space $\overline{\mathcal{M}}(x,y)$ is the <u>space of unparametrized trajectories</u>, and it is this which can be compactified.

To compactify such a space means to find a way to take limits of sequences in that space.

Geometry suggests that flow lines $x \to y$ may converge to broken flow lines $x \to z \to y$, where z is some critical point with $f(y) < f(z) < f(x)$



We want to make this notion of convergence precise. So we consider a sequence $(\overline{u}_n)_{n=0}^{\infty} \subset \overline{\mathcal{M}}(x,y)$

$\overline{\mathcal{M}}(x,y)$ is a quotient, so we represent $\overline{u}_n$ by $u_n \in \mathcal{M}(x,y)$

Now, u_n is only determined up to $\mathbb{R}$-action $u_n \sim u_n \cdot \tau$

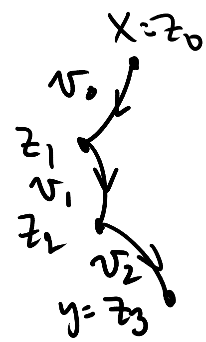
The potential limit is a <u>broken trajectory</u>: a configuration of

- critical points $x = z_0, z_1, \dots, z_r = y$

$$f(x) > f(z_1) > \dots > f(y)$$

- Trajectories $\overline{v}_j \in \overline{\mathcal{M}}(z_j, z_{j+1})$

$(j = 0, \dots, r-1)$



The notion of convergence is as follows

<u>Def</u> $(\overline{u_n})_{n=0}^{\infty}$ <u>converges geometrically</u> to $(\overline{v}_0, \dots, \overline{v}_{r-1})$

If there exist parametrized representatives u_n, v_j

And there exist sequences $\tau_{n,j} \in \mathbb{R}$ $\quad n = 0, \dots, \infty$, $\; j = 0, \dots, r-1$

Such that, for each j, the sequence $u_n \cdot \tau_{n,j}$

converges to v_j in the C^∞_{loc} sense

$$u_n \cdot \tau_{n,j} \xrightarrow{C^\infty_{loc}} v_j$$

(C^∞_{loc} means that, on any compact subset of the domain $\mathbb{R}$, the function and all derivatives converge <u>uniformly</u>)

Note: This notion of convergence is manifestly insensitive to the choice of representatives u_n, v_j since if we change

$$\left\{ \begin{array}{l} u_n \to u_n \cdot \sigma_n \\ v_j \to v_j \cdot \rho_j \end{array} \right\} \quad \text{can compensate} \quad \tau_{n,j} \to \tau_{n,j} - \sigma_n + \rho_j$$

Now we claim that $\overline{\mathcal{M}}(x,y)$ can be compactified by taking such limits:

<u>Proposition</u>: Any sequence $(\overline{u}_n)_{n=0}^{\infty} \subset \overline{\mathcal{M}}(x,y)$ contains a subsequence that converges geometrically to a broken trajectory with $\le k$ components, where $k = \mathrm{ind}(x) - \mathrm{ind}(y)$

<u>Lemma</u> Every sequence $(u_n)_{n=0}^{\infty} \subset \mathcal{M}(x,y)$ posesses a subsequence u_{n_k} such that

$$u_{n_k} \xrightarrow{C^\infty_{loc}} v \in C^\infty(\mathbb{R}, M)$$

<u>Proof</u>: First we claim C^0_{loc} convergence, and we intend to use Arzela-Ascoli. For this we need to show that (u_n) is

(i) pointwise bounded

(ii) equicontinuous:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall n)(d(s,t) < \delta \Rightarrow d(u_n(s), u_n(t)) < \varepsilon)$$

Note the order of logical quantifiers.

(i) follows trivially because the target M is compact
(More generally, we would need to use the so-called Palais–Smale condition for this step)

(ii) Now we estimate

$$\int_s^t |\dot{u}_n(\tau)|^2 d\tau = \int_s^t \langle \dot{u}_n, -\nabla f \circ u_n \rangle d\tau$$

$$= -\int_s^t u_n^* df = -(f(u_n(t)) - f(u_n(s))) \leq f(x) - f(y) \quad \forall s \leq t$$

Now $d(u_n(t), u_n(s)) \leq \int_s^t |\dot{u}_n(\tau)| d\tau$

$$\leq \left(|t-s| \int_s^t |\dot{u}_n(\tau)|^2 d\tau \right)^{1/2}$$

Cauchy-Schwarz $\int f \cdot 1 \leq \|f\|_2 \|1\|_2$

$$\leq \sqrt{t-s} \sqrt{f(x) - f(y)}$$ by previous estimate

So $(u_n)_{n=0}^{\infty}$ is equicontinuous

When u_n is restricted to any compact interval $[-R, R] \subset \mathbb{R}$, Arzela–Ascoli extracts a subsequence that converges uniformly on $[-R, R]$ to a continuous v.

Then in fact $\dot{v} = -\nabla f \circ v$ and v is smooth.
(Bootstrap)

Lemma: The v so obtained satisfies

$$\lim_{t \to \pm\infty} \nabla f(v(t)) = 0$$

and so $v \in \mathcal{M}(x', y')$ for some x', y'
$f(y) \le f(y') \le f(x') \le f(x)$

Proof $\quad u_n \xrightarrow{C^0_{loc}} v \quad$ and $\dot{v} = -\nabla f \circ v$

implies

$$\int_{-T}^{T} |\dot{v}(s)|^2 ds = f(v(T)) - f(v(-T)) \le f(x) - f(y)$$

Thus $\displaystyle\int_{-\infty}^{\infty} |\dot{v}|^2 ds < \infty$ is convergent

Hence $\displaystyle\lim_{t \to \pm\infty} \dot{v} = 0$ and so $\displaystyle\lim_{t \to \pm\infty} -\nabla f \circ v$ $\boxed{/}$

Now we construct various subsequences...

Why is topology of "geometric convergence" Hausdorff
That is, why can't a sequence $(\bar{u}_n)_{n=0}^{\infty}$ have two geometric limits $(\bar{v}_0, \dots, \bar{v}_{r-1})$ and $(\bar{v}_0', \dots, \bar{v}_{r'-1}')$?

Because it really is a geometric notion of convergence.
Recall The <u>Hausdorff distance</u>: (X, d) a metric space

Let $\mathcal{K} = \{ K \subset X \mid K \text{ is compact} \}$

For $A, B \in \mathcal{K}$,

let $$d_H(A,B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a,b) \, , \, \sup_{b \in B} \inf_{a \in A} d(a,b) \right\}$$

This is a metric on $\mathcal{K}$, in particular $d_H(A,B) = 0 \Rightarrow A = B$
Hence the topology induced by d_H satisfies the Hausdorff separation axiom.
("The Hausdorff topology is Hausdorff", haha)

For any flow line $u \in \mathcal{M}(x,y)$, we obtain a compact subset by taking the image and including the endpoints.

$$(u)^{cl} := u(\mathbb{R}) \cup \{x, y\}$$

Similarly a broken flow line $v = (v_0, \dots, v_{r-1})$ yields the compact subset:
$$(v)^{cl} = \bigcup_i (v_i)^{cl}$$

Now we will show

<u>Proposition</u> If $(u_n)_{n=0}^{\infty}$ converges geometrically to $v = (v_0, \dots, v_{r-1})$

then $(u_n)^{cl} \xrightarrow{d_H} v^{cl}$, i.e. $d_H((u_n)^{cl}, (v)^{cl}) \to 0$

that is the map $w \to (w)^{cl}$ is continuous.

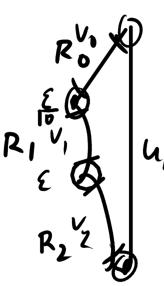
<u>Proof</u>: We know $u_n \cdot \tau_{n,j} \xrightarrow{C^0_{loc}} v_j$

Thus $(u_n \cdot \tau_{n,j})([-R,R]) \xrightarrow{d_H} v_j([-R,R])$

So $\sup_{a \in v_j[-R,R]} \inf_{b \in (u_n)^{cl}} d(a,b) \longrightarrow 0$

This holds for all j and R

Fix $\varepsilon > 0$
pick $\frac{\varepsilon}{2}$ balls at crit pts of v,
R_i such that complement is
covered by $v_i[-R_i, R_i]$



We conclude that eventually

$$v_j[-R_j, R_j] \subset B_{\frac{\varepsilon}{2}}((u_n)^{cl})$$

Then $(v)^{cl} \subset B_\varepsilon((u_n)^{cl})$

Also need to show that eventually $(u_n)^{cl} \subset B_\varepsilon(v^{cl})$

<u>Suppose not</u>: Then $\exists$ subseq $u_{n_k}, t_k \in \mathbb{R}$ such that $u_{n_k}(t_k) \notin B_\varepsilon(v^{cl})$

Then $f(u_{n_k}(t_k))$ is a sequence in the interval $[f(y), f(x)]$

So pass to subseq $\Rightarrow f(u_{n_k}(t_k))$ converges. to some a

Use lemma to get subsequence such that $(u_{n_k} \cdot t_k) \xrightarrow{C^\infty_{loc}} w \in \mathcal{M}(p,q)$

Then $f((u_{n_k} \cdot t_k)(0)) \longrightarrow a$

so so $w(0) \in f^{-1}(a)$

But now, our geometric limit $v = (v_0, \dots, v_{r-1})$ has some bit crossing $f^{-1}(a)$, say v_j assume $f(v_j(0)) = a$

$$u_n \cdot \tau_{n,j} \xrightarrow{C^\infty_{loc}} v_j$$

Thus $(u_n \cdot \tau_{n,j})(0) \longrightarrow v_j(0) \in f^{-1}(a)$

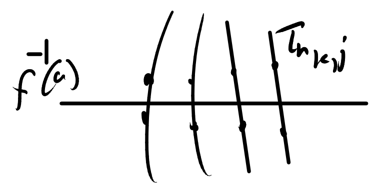
But what we have assumed for a contradiction, w is not a reparametrization of v_j, since by construction $w(0) \notin B_\varepsilon(v^{cl})$.

$$\text{So } (u_{n_k} \cdot \tau_{n_k,j})(0) \longrightarrow v_j(0) \in f^{-1}(a)$$

$$(u_{n_k} \cdot t_k)(0) \longrightarrow w(0) \in f^{-1}(a)$$

$$\text{Then } u_{n_k}(\tau_{n_k,j}) \longrightarrow v_j(0) \in f^{-1}(a)$$

$$u_{n_k}(t_k) \longrightarrow w(0) \in f^{-1}(a)$$



Suppose a is not a critical value of f:

Then $|\nabla f|$ and hence $|\dot{u}_{n_k}|$ is bounded away from zero along the sequences we are interested in.

Then since $d\left(f\left(u_{n_k}(t_k)\right), f\left(u_{n_k}(\tau_{n_k,j})\right)\right) \xrightarrow{k\to\infty} 0$

$$\Vert$$

$$\left| \int_{t_k}^{\tau_{n_k,j}} |\dot{u}_{n_k}(s)|^2 ds \right| \geq C \cdot | t_k - \tau_{n_k,j} |$$

Thus $|t_k - \tau_{n_k,j}| \to 0$

On the other hand for arb small η, k suff large.

$$d\left(u_{n_k}(t_k), u_{n_k}(\tau_{n_k,j})\right) \geq \underbrace{d(w(0), v_j(0))}_{\text{big}} - \eta$$

$$/\!\wedge$$

$$\int_{t_k}^{\tau_{n_k,j}} |\dot{u}_{n_k}(s)| ds \leq \underbrace{\sqrt{|t_k - \tau_{n_k,j}|}}_{\text{Take this small, get contradiction.}} \cdot \sqrt{f(x) - f(y)}$$

But we assumed a was not a critical value of f. If we choose the original sequence carefully enough, we can achieve this.

Now it remains to actually characterize the boundary strata in our "Morse-Smale-Floer" compactification.

We need to see that every broken flow line occurs in the limit of some sequence of smooth flow lines.

In particular, in the case $\mathrm{ind}(x) - \mathrm{ind}(y) = 2$ we need to show that $(\overline{\mathcal{M}}(x,y))^\wedge$ is a 1-dimensional manifold with boundary.