Invariance of Floer homology in the exact case
Application to cotangent bundles.

For this lecture we assume that (M, ω) is exact and we fix a choice of 1-form θ such that $\omega = d\theta$

With respect to the choice of θ, a Lagrangian L is exact if $\theta|_L = df_L$ for some function $f_L : L \to \mathbb{R}$

The exactness assumptions imply the asphericity conditions
$\langle \omega, \pi_2(M) \rangle = 0 \quad \langle \omega, \pi_2(M, L) \rangle = 0$
So the arguments from the last lecture show that $\partial \circ \partial = 0$ on the Floer complex $CF(L_0, L_1)$ if L_0 and L_1 are exact.

The main example for today: $\quad M = T^*Q \qquad \theta = \sum -p_i dq_i$
$\omega = \sum dq_i \wedge dp_i$

$L_0 = Q$ (0-section) $\qquad L_1 = \phi_H^1(Q)$, where

ϕ_H^1 is the time-1 flow of a Hamiltonian $H : M \to \mathbb{R}$

<u>Thm</u> (Floer) $\qquad HF(Q, \phi_H^1(Q)) \simeq H^*(Q; \mathbb{Z}_2)$

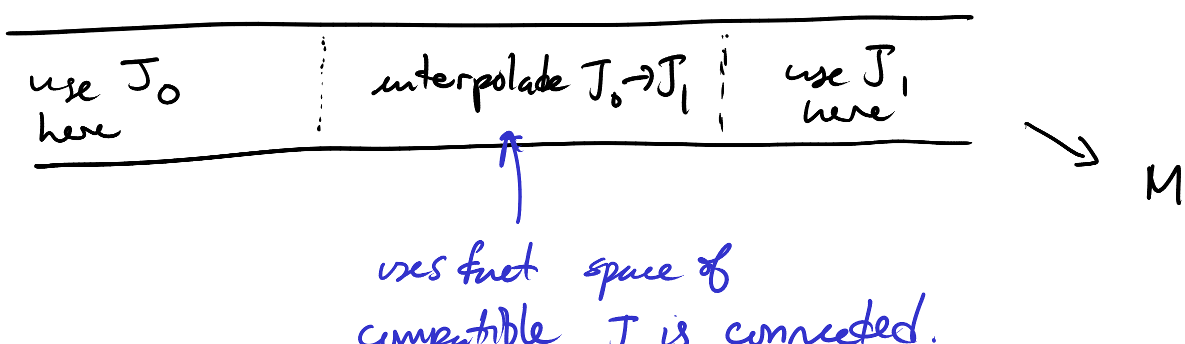
The proof relies on
(1) Invariance of HF with respect to change in complex structure
(2) Invariance of HF with respect to change of H.
(3) Computation of $HF(Q, \phi_H^1(Q))$ for particular convenient choices of J and H.

To prove invariance: Continuation maps and homotopy operators.

Assume J_0 and J_1 are two almost complex structures for which the Floer complex $CF(L_0,L_1)$ can be defined. Call the corresponding Floer boundary operators $\partial(J_i)$

We want to show $(CF(L_0,L_1), \partial(J_0))$ is chain homotopy equivalent to $(CF(L_0,L_1), \partial(J_1))$

We define a chain map by looking at strips with J depending on the point in the domain.



Define Continuation map $\mathcal{X}_{0,1} : (CF(L_0,L_1), \partial(J_0)) \to (CF(L_0,L_1), \partial(J_1))$ By counting 0-dimensional components of the moduli space

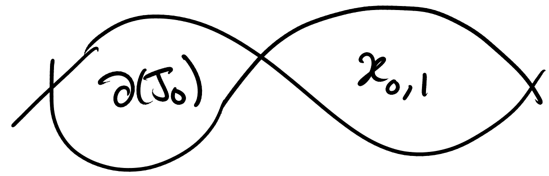
$$\mathcal{X}_{0,1}(x) = \sum_y \# \overline{\mathcal{M}}(\{J_s\}_{s\in\mathbb{R}})^0 \, y$$

Note that, unlike ∂, there is no $\mathbb{R}$-symmetry, so we can expect to get 0-dimensional components.

Why is $\mathcal{X}_{0,1}$ a chain map?

Consider 1-dimensional components of $\overline{\mathcal{M}}(\{J_s\}_{s\in\mathbb{R}})$

The boundary is



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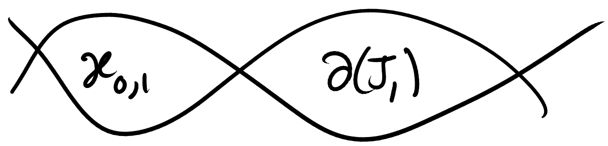


So $\partial(J_1)\circ\mathcal{X}_{0,1} + \mathcal{X}_{0,1}\circ\partial(J_0) = 0$

Swapping roles of J_0 and J_1, we get a continuation map $\mathcal{X}_{1,0} : (CF(L_0,L_1), \partial(J_1)) \to (CF(L_0,L_1), \partial(J_0))$

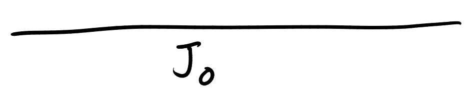
We want to show $\mathcal{X}_{0,1}\circ\mathcal{X}_{1,0}$ and $\mathcal{X}_{1,0}\circ\mathcal{X}_{0,1}$ are homotopic to identity maps. This means finding a map $P \circlearrowright (CF(L_0,L_1), \partial(J_0))$ such that

$$\partial(J_0)\circ P + P\circ\partial(J_0) = Id - \mathcal{X}_{1,0}\circ\mathcal{X}_{0,1}$$

For this we consider a 1-parameter family of 1-parameter families of $\{J_s\}$.

Let $R\in[0,\infty)$ be the new parameter.

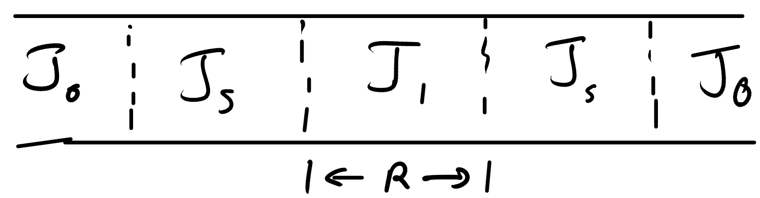
At $R=0$ $\quad J\equiv J_0$ constantly



As R increases to 1 the constant path J_0 is deformed to a path that interpolates from J_0 to J_1 via the same $\{J_s\}$ used to define $x_{0,1}$, then back to J_0 via the path used to define $x_{1,0}$.

Uses $\mathcal{J}_\omega$ is 1-connected

$R \geq 1$



Then as $R \to \infty$, the region where $J = J_1$ increases in length without bound.

This defines a 1 parameter family of strip counting problems parametrized by $R \in [0, \infty)$

Define P by counting 0-dimensional components of the parametrized moduli space. This means we count "exceptional" strips: those that a rigid even with respect to the variation of the R-parameter.

In the graded situation, where ∂ has degree -1, P will have degree $+1$.

Now to prove the homotopy formula

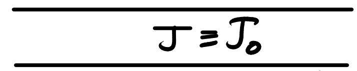
$$\partial P + P \partial = Id - x_{1,0} \circ x_{0,1}$$

we use the Gromov-Compactification of the 1-dimensional components of the R-parametric moduli space.

The Gromov boundary consists of

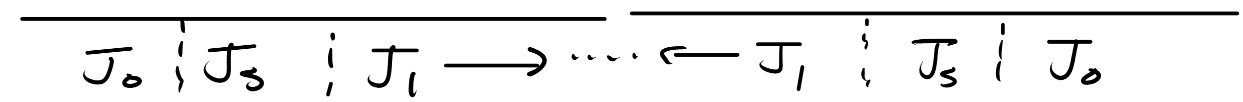
- $R = 0$ end
- $R = \infty$ end
- Floer differential strip breaking

The $R=0$ end counts



strips with constant $J \equiv J_0$, like the differential. But since we are looking at index 0 strips, they must be constant. Thus the $R=0$ end contributes the identity map Id.

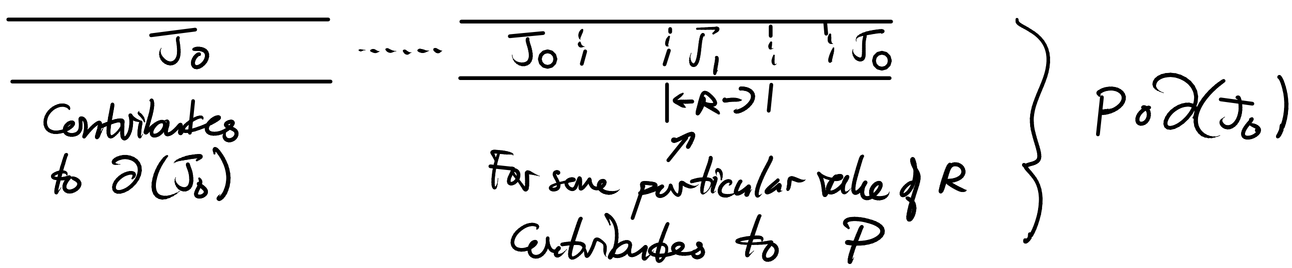
At the $R=\infty$ end, we have compactified by adding a stratum corresponding to pairs of strips

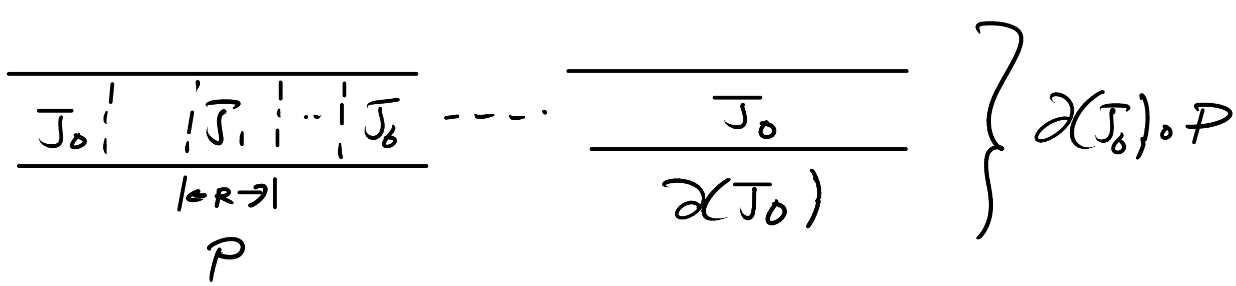


This is precisely the moduli space defining

$$\mathcal{K}_{1,0} \circ \mathcal{K}_{0,1}$$

Floer differential breaking will result in boundaries consisting of a Floer differential (constant J) joined to an exceptional strip in the R-family





Counting the boundary points modulo 2, we obtain

$$Id + \mathcal{K}_{1,0}\circ\mathcal{K}_{0,1} + \partial(J_0)\circ P + P\circ\partial(J_0) = 0 \quad \text{as desired}$$

Swapping the roles of J_0 and J_1, we get a homotopy between Id and $\mathcal{K}_{0,1}\circ\mathcal{K}_{1,0}$.

Thus both $\mathcal{K}_{0,1}$ and $\mathcal{K}_{1,0}$ are chain homotopy equivalences

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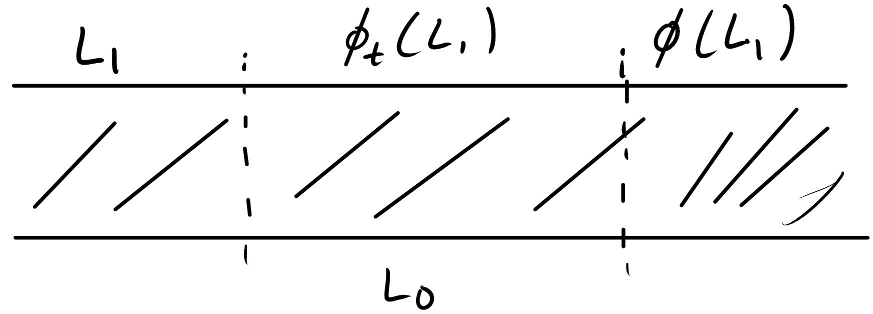
We can use a similar continuation map argument to show that $HF(L_0,L_1)$ is invariant under Hamiltonian deformation of L_0 or L_1. Let ϕ be the Hamiltonian diffeomorphism generated by a time-dependent Hamiltonian H_t. We assume that $L_0 \pitchfork L_1$ and $L_0 \pitchfork \phi(L_1)$.

Then $\phi = \phi_1$, where $\{\phi_t\}_{t\in[0,1]}$ is the isotopy generated by the time-dependent vector field X_t

$$X_t : \quad \omega(-, X_t) = dH_t$$

The sort of "continuation strips" we look at now have a fixed almost complex structure J, but a moving Lagrangian boundary condition along the edge corresponding to L_1



$$\mathcal{K}_\phi : (CF(L_0,L_1), \partial(J)) \longrightarrow (CF(L_0,\phi(L_1)), \partial(J))$$

There is similarly $\mathcal{H}_{\phi^{-1}} : CF(L_0, \phi(L_1)) \to CF(L_0, L_1)$

Similar arguments as before show that
$\mathcal{H}_\phi$ & $\mathcal{H}_{\phi^{-1}}$ are chain maps that are mutually homotopy-inverse homotopy equivalences.

Now we want to compute the "self-Floer homology"
$$HF(L,L) \quad \text{for } L \subset T^*L$$
Since L and L are not transverse, this symbol has no meaning so far. But we can define it to be
$$HF(L, \phi(L))$$
where $\phi(L)$ is a transverse hamiltonian push-off of L, and the group is defined using any regular J. This is meaningful because we now know that the result is independent of these choices.

Moreover, since we can take any ϕ and J we want (as long as they satisfy the regularity conditions) the game is now to find a particularly clever choice that allows us to compute.

Here we follow Floer, "Witten's complex and ∞-dim'l Morse theory"
To start choose a metric g on L and a Morse function
$$f : L \to \mathbb{R}$$
We assume (g, f) is Morse-Smale, so the morse complex is defined. $C_*^{\text{morse}}(L, f) \circlearrowleft \partial(g, f)$

On T^*L, These same data (g, f) give us a hamiltonian pushoff of L, and an almost complex structure as follows:

The metric g on L induces a metric $\tilde{g}$ on T^*L

Such that the splitting $T(T^*L) \simeq T^{vert}(T^*L) \oplus T^{horiz}(T^*L)$ is $\tilde{g}$-orthogonal, where $T^{horiz}(T^*L)$ is defined by the Levi-Civita connection of g and

$$T_x^{vert}(T^*L) = T^*_{\pi(x)}L \quad \text{and} \quad T_x^{horiz}(T^*L) \simeq T_{\pi(x)}L$$

have metrics induced by g.

Let $H = f \circ \pi$ (time-independent), ϕ_t its flow.
Then $\phi_1(L) = \text{graph } df$. Let $\circ$

Define $J_t = (\phi_t)_* J (\phi_t)_*^{-1}$

We consider $CF(L, \phi_1(L)) \circlearrowleft \partial(\{J_t\}_{t \in [0,1]})$

- Generators $\overset{1:1}{\longleftrightarrow}$ critical points of f

J depends on t-coordinate on $(s,t) \in \mathbb{R} \times [0,1]$

- There is a relationship between gradient flow lines for the morse smale pair (g, f) on L and J_t-strips in this Floer complex.

$\gamma : \mathbb{R} \to L \qquad \frac{d}{ds}\gamma(s) + (\text{grad}_g f)(\gamma(s)) = 0$

$u(s,t) = \varphi_t(\gamma(s)) \qquad \frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} = 0.$

Floer shows that if f is C^2-small, this is a bijection between flow lines and strips.

Thus $\left(CF(L, \phi_1(L)), \partial(\{J_t\})\right) \simeq \left(C_*^{morse}(L,g), \partial(g,f)\right)$ are isomorphic chain complexes.

Thus $HF(L, \phi_1(L)) \simeq H_*^{morse}(L) \simeq H_*(L)$