Index theorems for Pseudoholomorphic curves

We know that the inverse image of a regular value of a Fredholm map is a smooth manifold whose dimension is the index of the linearization $\mathrm{ind}(D_u \bar{\partial}_J)$.

But what <u>is</u> this number? The index theory of Atiyah-Singer provides a framework for answering such questions.

Background: we saw that $\mathrm{ind}(D)$ is unchanged by norm-continuous deformations of D staying within the space of Fredholm operators. It was observed that homotopy of the geometric data defining D (e.g. change in J, metric, etc.) therefore do not change the index. So Gelfand proposed to find a topological formula for $\mathrm{ind}(D)$ in terms of the geometry of the problem.

Atiyah-Singer solved this. The result could also be cast as a vast generalization of the Riemann-Roch theorem.

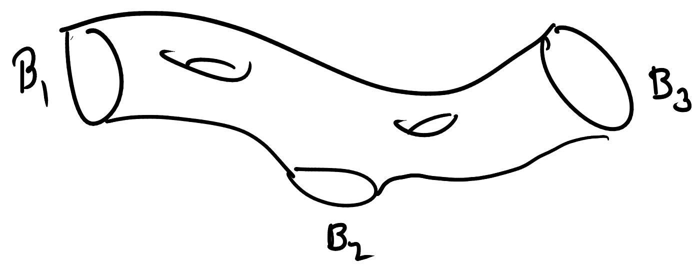
<u>Thm</u> (Riemann-Roch) Let (S, j) be a compact Riemann Surface $u: S \longrightarrow M$ a map to the almost complex manifold (M, J). The index of the linearized operator $D_u \bar{\partial}_J$ is

$$\mathrm{ind}_{\mathbb{R}}(D_u \bar{\partial}_J) = (\dim_{\mathbb{C}} M)\, \chi(S) + 2 \langle c_1(TM), u_*[S] \rangle$$

(If $D_u \bar{\partial}_J$ is $\mathbb{C}$-linear: $\mathrm{ind}_{\mathbb{C}} = (\dim_{\mathbb{C}} M)(1-g) + \deg(u^* TM)$)

We want a generalization that will cover holomorphic curves with boundary on Lagrangian submanifolds, and also with asymptotic conditions at intersection points

Assume S is compact with boundary $\partial S = \coprod_i B_i$



and each boundary B_i is mapped to a Lagrangian L_i

The linearized operator at $u: S \to M$ acts on sections of $u^* TM$ satisfying the restriction that along B_i the section must lie in $u^* TL_i$

Thus we have a complex vector bundle $u^* TM$ over S with subbundles $u^* TL_i$ along B_i

Since S is <u>not</u> closed, $u^* TM$ can be <u>trivialized</u> over S. But the subbundles $u^* TL_i$ "go along for the ride", we <u>cannot</u> trivialize them simultaneously.

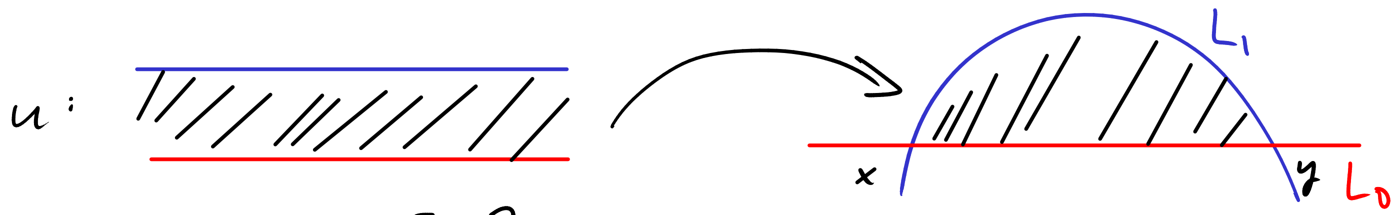
The nontriviality of the subbundle $u^* TL_i$ is measured by the Maslov index $\mu: \pi_1(U_n/O_n) \xrightarrow{\sim} \mathbb{Z}$

Each B_i with its induced boundary orientation determines a loop in the Lagrangian grassmannian of $\mathbb{C}^n$, where we trivialize $u^* TM \cong S \times \mathbb{C}^n$.

<u>Thm</u> (Riemann-Roch with boundary) In the set up just described

$$\mathrm{ind}_{\mathbb{R}}(D_u \bar{\partial}_J) = (\dim_{\mathbb{C}} M)\,\chi(S) + \sum_i \mu(B_i)$$

Now we consider the case of strips



$S = \mathbb{R} \times [0,1]$
$(s\ ,\ t)$

The boundary is not a loop, and the image has "corners" at intersection points $x, y \in L_0 \cap L_1$.

We will generalize the Maslov index to this situation. Consider the complex vector bundle u^*TM over S. Once again we may trivialize it. Thus we get subbundles u^*TL_0 over $\mathbb{R} \times \{0\}$ and u^*TL_1 over $\mathbb{R} \times \{1\}$.

We assume that

non degenerate critical pt of action!

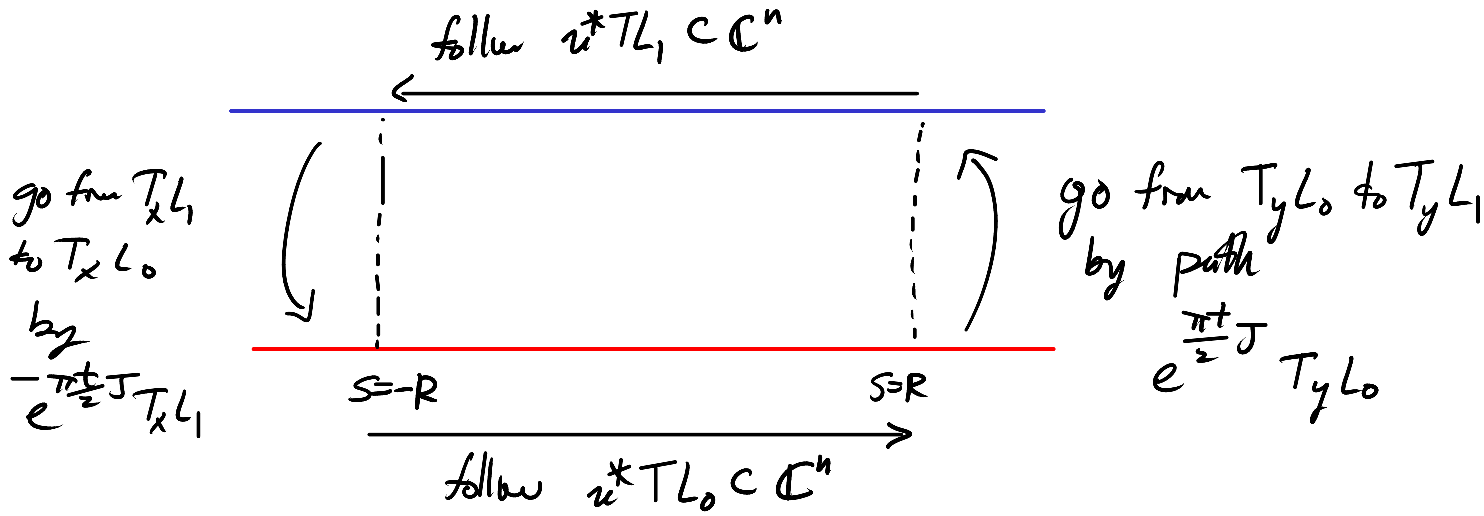
(i) L_0 and L_1 intersect transversely at x and y

(ii) The complex structure J on M is such that

$$J \cdot T_x L_0 = T_x L_1 \quad \text{in } T_x M$$
$$J \cdot T_y L_0 = T_y L_1 \quad \text{in } T_y M.$$

(iii) For $|s| > R$, the subbundles u^*TL_0 and u^*TL_1 are constant w.r.t. the chosen trivialization of u^*TM.

Now we define a loop in the Lagrangian Grassmannian.



The Maslov index $\mu(x,y,u)$ is μ of this loop.

Floer's Riemann-Roch theorem is

<u>Thm</u> In the above situation: $D_u\bar{\partial}_J$ has a Fredholm extension

$$L^2_1(S, u^*TM, u^*TL_0, u^*TL_1) \to L^2\text{-}\Omega^{0,1}(S, u^*TM)$$

and

$$\operatorname{ind}(D_u\bar{\partial}_J) = \mu(x,y,u)$$

<u>Remarks</u> Assumption (i) is necessary for the Fredholm property (otherwise use weighted Sobolev spaces)

The term $(\dim_{\mathbb{C}} M)\,\chi(S)$ is not present because of the convention defining $\mu(x,y,u)$

<u>Dependence on the homotopy class of u:</u>

In general, $\mu(x,y,u)$ does not depend only on x and y

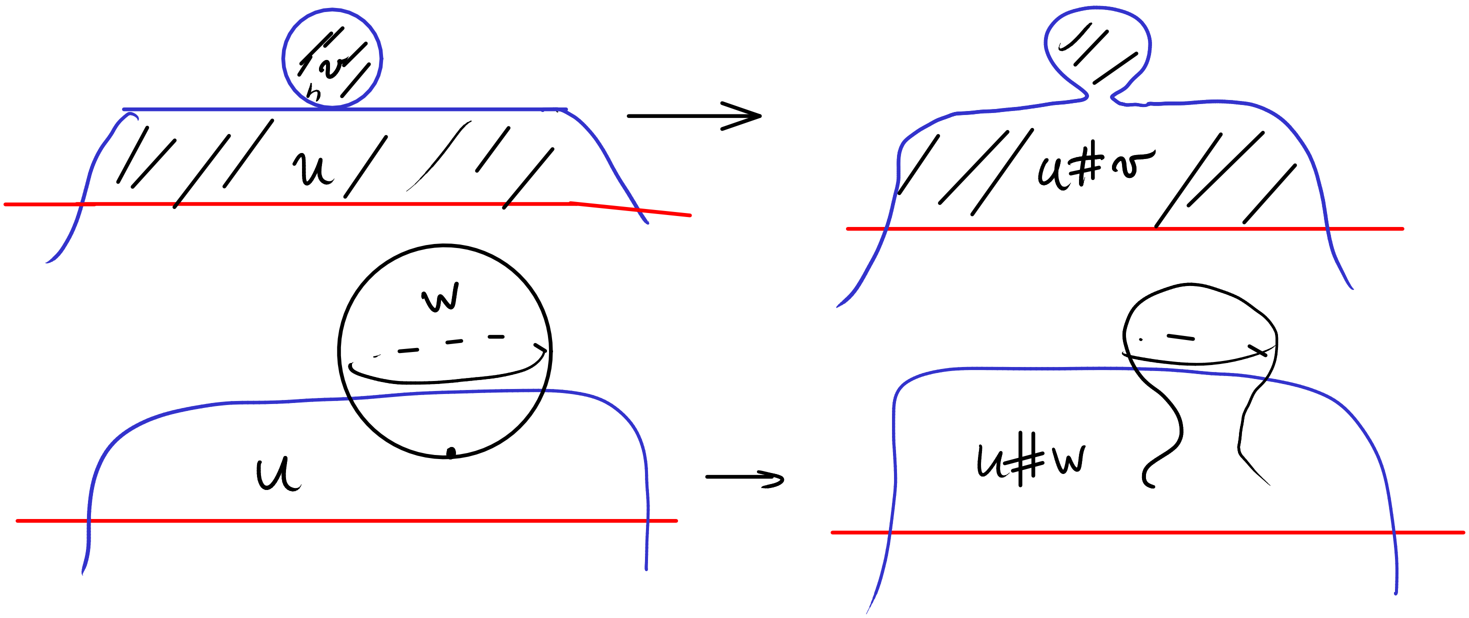
For example, if there are homotopically nontrivial maps

$$v : (D^2, S^1) \to (M, L)$$

or

$$w : S^2 \longrightarrow M$$

Then taking connected sum of u with such surfaces will change the homotopy class of u, and (possibly) the index.



Using a linear gluing analysis, one can show that these connected sum operations change the index by the formulas

Disk v : $\mu(x, y, u\#v) = \mu(x,y,u) + \mu(v)$

Sphere w : $\mu(x, y, u\#w) = \mu(x,y,u) + 2\langle c_1(TM), w_*[S^2]\rangle$

Consequences for the grading on Floer homology $HF^*(L_0, L_1)$

The differential counts strips in classes such that $\mu(x,y,u) = 1$. To make $CF^*(L_0,L_1)$ $\mathbb{Z}$-graded would need to introduce function $\mu(x)$ such that

$$\mu(x,y,u) = \mu(x) - \mu(y)$$

The ambiguity makes this impossible, unless we put in some conditions on $c_1(TM)$ and $\mu(v)$ for $v:(D^2,S^1)\to(M,L_i)$

Eg assume $2c_1(TM)=0$ (similiar to Calabi-Yau)
This means that there is a well defined homomorphism
$\mu_L : H_1(L)\to\mathbb{Z}$ such that (Maslov class of L)
$$\mu(v)=\mu_L(\partial v) \quad \text{for any disk } v.$$

Then assume $\mu_{L_0}=0$ and $\mu_{L_1}=0$ $\Rightarrow$ get $\mathbb{Z}$-graded Floer homology.

In general we can only get a grading in $\mathbb{Z}/N\mathbb{Z}$ where $N\mathbb{Z}$ is the subgroup generated by the ambiguity terms $2\langle c_1(TM), w_*[S^2]\rangle$ and $\mu(v)$

If L_0 and L_1 are oriented, $\mu(v)\in 2\mathbb{Z}$ $\forall v$
of course $2\langle c_1(TM), w_*[S^2]\rangle\in 2\mathbb{Z}$
So $2|N$ in this case, and Floer homology $CF_*(L_0,L_1)$ is at least $\mathbb{Z}/2\mathbb{Z}$-graded.