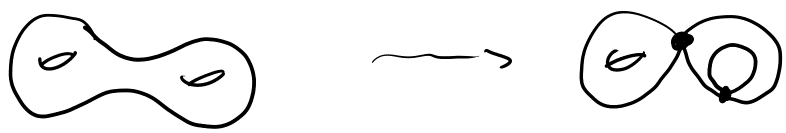
More about Gromov Compactness

Gromov compactness (and its converse, gluing) is the fundamental tool to establish the algebraic structure of Floer homology.

Gromov Compactness: Pseudo-holomorphic maps degenerate to stable maps.

- Compact Riemann surfaces (with marked points) ⇝ Nodal curves: Deligne-Mumford $\overline{\mathcal{M}}_{g,n}$
  stable = discrete automorphisms

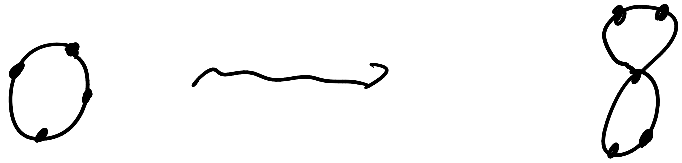


HW: Describe $\overline{\mathcal{M}}_{0,3}$, $\overline{\mathcal{M}}_{0,4}$, $\overline{\mathcal{M}}_{0,5}$, $\overline{\mathcal{M}}_{1,1}$

- Maps from compact RS to X (with marked points) Modulo reparametrization ⇝ Maps from nodal RS. $\overline{\mathcal{M}}_{g,n}(X,\beta)$ $\beta \in H_2(X)$
  Stable = discrete automorphisms modulo reparametrization

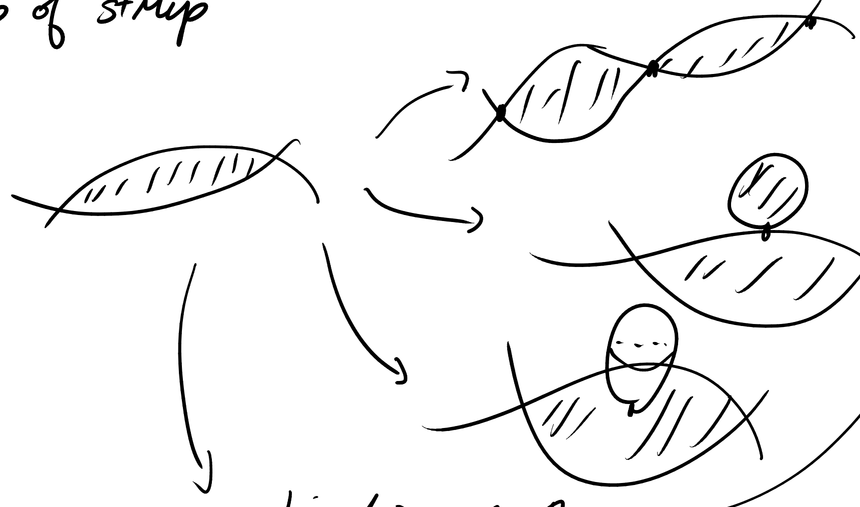
Homework: describe $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 1)$, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$, $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2, 3)$

- Disks with boundary Marked points ⇝ Nodal disks: Stasheff space $\overline{R}_d$ (Associahedron)
  HW: justify this in terms of "parenthesizations".



- map of disk to (M,L) ⇝ Stable maps of disks
  Can now have Sphere bubbles!

• map of strip



The moral for now is that, unless we make some restrictions on what we consider, the boundary strata can be extremely complicated, which makes transversality (and gluing) difficult. (Which we need for the algebraic structure)

The thing that does control the complexity of the degenerations is the symplectic form ω. If J is compatible with ω, then ω is positive on any pseudoholomorphic curve. The symplectic area of a J-curve C is $\int_C \omega > 0$

The stable map compactification is always set up so that symplectic area is preserved in limits. One shows that the area of a nonconstant curve is bounded away from zero (depending on M, ω, L, etc). This implies only finitely many strata in the compactification.

We can also use this idea to preclude "bad degenerations" a priori. Call pair (M,L) <u>aspherical</u> if $\langle \omega, \pi_2(M,L)\rangle = 0$ i.e. $\int_{D^2} u^*\omega = 0 \quad \forall\ u:(D^2,S^1)\to(M,L)$. This implies $\langle \omega, \pi_2(M)\rangle = 0$.

This property is implied by exactness of M and L

(M,ω) exact $\Longleftrightarrow$ $\omega = d\theta$ $\quad \theta \in \Omega^1(M)$

$\Rightarrow \int_{S^2} u^*\omega = 0$ for any $u: S^2 \to M$

$L \subset (M, \omega = d\theta)$ exact $\Longleftrightarrow$ $\theta|_L = df$ $\quad f \in C^\infty(L)$

$$\Rightarrow \int_{D^2} u^*\omega = \int_{S^1} u^*\lambda = 0$$

Note that M cannot be compact if it is exact. This means a little extra care must be taken to prove gromov compactness.

A Compact aspherical example: $\mathbb{R}^{2n}/\Lambda$ $\quad \Lambda$ full rank lattice with a linear lagrangian $L \subset \mathbb{R}^{2n}/\Lambda$

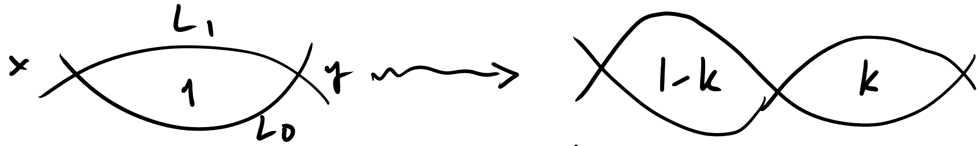
(This is one of the classical cases for the Arnold conjecture)

<u>Assume</u> (M, L_0, L_1) are aspherical

Defining $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$

Count index 1 strips modulo $\mathbb{R}$-translation
Finiteness follows from Gromov compactness

asphericity implies strips in $\overline{\mathcal{M}}(x,y)$ have same area.



No other degenerations by asphericality. Either $1-k$ or $k < 1$
No such strip can be transverse $\Rightarrow$ don't exist for generic J.

Thus $\overline{\mathcal{M}}(x,y)^1/\mathbb{R}$ is compact smooth manifold of dim 0
Count points to define ∂ (mod 2, or with orientations if we set that up)

$$\partial x = \sum_y \#\left(\overline{\mathcal{M}}(x,y)^1/\mathbb{R}\right) y$$

Now prove its a complex $\partial^2 = 0$

$$\partial\partial x = \sum_z \sum_y \#\left(\overline{\mathcal{M}}(x,y)^1/\mathbb{R}\right)\#\left(\overline{\mathcal{M}}(y,z)^1/\mathbb{R}\right) z$$

Plan: show that $\left(\overline{\mathcal{M}}(x,y)^1/\mathbb{R}\right)\times\left(\overline{\mathcal{M}}(y,z)^1/\mathbb{R}\right)$ is null cobordant.

It's the boundary of $\overline{\mathcal{M}}(x,z)^2/\mathbb{R}$

index 2 component.

Reason