Grading on Lagrangian Floer cohomology [Seidel]

We have already made some ad hoc grading arguments in previous lectures, but for the next set of applications it will be important to do this more systematically.

Recall the Lagrangian grassmannian for (V,β) symplectic V.S.

$$\Lambda(V,\beta) = \{\text{Lag subspaces in } V\} \simeq U(n)/O(n)$$
$$\dim V = 2n$$

Recall $\pi_1(\Lambda(V,\beta)) \simeq H_1(\Lambda(V,\beta)) \simeq H^1(\Lambda(V,\beta)) \simeq \mathbb{Z}$

Covers of X with deck group $\mathbb{Z}/N\mathbb{Z}$ are classified by $H^1(X;\mathbb{Z}/N)$

Let $C(V,\beta)$ denote the generator of $H^1(\Lambda(V,\beta);\mathbb{Z})$
Then the N-fold Maslov cover $\Lambda^N(V,\beta)$ is associated to the image of $C(V,\beta)$ in $H^1(\Lambda(V,\beta);\mathbb{Z}/N)$. It is also the connected cover that corresponds to the subgroup $N\mathbb{Z} \subset \mathbb{Z} \simeq \pi_1(\Lambda(V,\beta))$

The $\mathbb{Z}/N$ deck group action on Λ^N is denoted by ρ.

If $N=\infty$, we are talking about the universal covering $\tilde{\Lambda} = \Lambda^\infty$

Now consider a symplectic manifold (M,ω)
let $\Lambda \to M$ be the bundle of Lagrangian grassmannians
let $\Delta = \Lambda^n(TM,J)^{\otimes 2}$ be the square of the complex determinant of the tangent bundle (i.e. $K_M^{\otimes(-2)}$)

- An N-fold Maslov covering is a $\mathbb{Z}/N$-cover $\begin{array}{ccc} \Lambda^N & \to & \Lambda \\ & \searrow_M \swarrow & \end{array}$ whose restriction to $x \in M$ is isomorphic to $\Lambda^N(T_xM, \omega_x) \to \Lambda(T_xM, \omega_x)$

- A global mod-N Maslov class is $C^N \in H^1(\Lambda; \mathbb{Z}/N)$ whose restriction to any fiber is the mod-N reduction of $C(T_xM, \omega_x)$

- an N-th root of $\Delta = K_M^{\otimes(-2)}$ is a pair (Z, j) where Z is a complex line bundle and $j : Z^{\otimes N} \to \Delta$ is an iso. Declare $(Z_1, j_1) \sim (Z_2, j_2)$ if $\exists\, r : Z_1 \xrightarrow{\sim} Z_2$ s.t. $j_1 = j_2 r^{\otimes N}$

If $N = \infty$, an ∞-root is a trivialization, equivalence = homotopy.

Prop $\{N\text{-fold maslov coverings}\}/\text{iso} \overset{1:1}{\longleftrightarrow} \{\text{Global mod } N \text{ maslov classes}\}$ $\overset{1:1}{\longleftrightarrow} \{N\text{-th roots of } \Delta\}/\sim$.

Prop: (M, ω) admits an N-fold Maslov covering iff $2c_1(M, \omega)$ goes to zero in $H^2(M; \mathbb{Z}/N)$. Isomorphism classes are then an affine space over $H^1(M; \mathbb{Z}/N)$

Now let $L \subset M$ be a Lagrangian submanifold.
There is a section $s_L : L \to \Lambda|_L$: $s(x) = T_xL \in \Lambda(T_xM, \omega_x)$

- An <u>N-grading</u> of L is a lift $s_L^N : L \to \Lambda^N|_L$
L is called <u>N-gradable</u> if this exists.

Prop: L is N-gradable iff $s_L^*(C^N) \in H^1(L; \mathbb{Z}/N)$ vanishes where C^N is the global maslov class of Λ^N.

The set of N-gradings on connected L is a $\mathbb{Z}/N$-torsor
In particular if $H^1(L,\mathbb{Z}/N)=0$, L is N-gradable.

<u>Automorphisms:</u> let $\phi : M \circlearrowleft$ be a symplectomorphism

the action of ϕ naturally lifts to $\begin{array}{c}\Lambda \\ \downarrow \\ M\end{array}$

An N-grading of ϕ is a lift $\tilde{\phi} \circlearrowright \Lambda^N$ compatible with $\phi \circlearrowright \Lambda$.

- Not every symplectomorphism is N-gradable, but the obstruction lives in $H^1(M;\mathbb{Z}/N)$

Relation to previous discussion: N_M = minimal chern number
= generator of $\langle c_1(M), H_2(M)\rangle \subset \mathbb{Z}$

N_L = minimal maslov number = generator of image of maslov hom
$\mu: H_2(M,L) \to \mathbb{Z}$.

<u>Prop</u>: Assume $H_1(M;\mathbb{Z})=0$.
M admits N-fold maslov covering iff $N \mid 2N_M$ (which is then unique)
L admits N-grading iff $N \mid N_L$

<u>Examples</u> Orientations Λ^2 = oriented Lagrangian grassmanian.
since $2c_1(M) \bmod 2 = 0 \in H^2(M;\mathbb{Z}/2)$ any (M,ω)
admits a 2-fold Maslov cover. The cover can be twisted
by any real line bundle ξ, with class $w_1(\xi) \in H^1(M,\mathbb{Z}/2)$

$L \subset M$ is 2-gradable iff L is orientable. It then admits
exactly two 2-gradings

$c_1(M)$ is two torsion. If $2c_1(M) = 0 \in H^2(M;\mathbb{Z})$

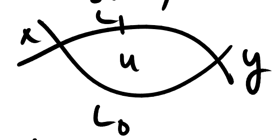
Then M admits an ∞-grading Λ^∞, can be twisted by $H^1(M;\mathbb{Z})$

L admits an ∞-grading iff $\mu : H_2(M,L) \to \mathbb{Z}$ vanishes identically.
(vanishing Maslov class)

Subexample: M = Calabi-Yau manifold, L = Lagrangian sphere.

Absolute $\mathbb{Z}/N$ grading on Floer homology

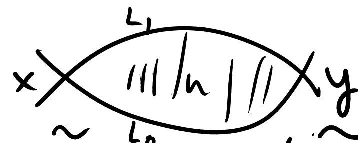
In our previous discussion, we associated a Maslov index to a strip u:



Now suppose L_0 and L_1 come with lifts $\tilde{S}_{L_i} : L_i \to \Lambda^N|_{L_i}$

Theorem To a pair of Lagrangians $\tilde{\ell}_0, \tilde{\ell}_1 \in \Lambda^N$, we can associate an index $\tilde{\mu}(\tilde{\ell}_0, \tilde{\ell}_1) \in \mathbb{Z}/N$

With the properties:
- $\tilde{\mu}(\rho(a)\tilde{\ell}_0, \rho(b)\tilde{\ell}_1) = \tilde{\mu}(\tilde{\ell}_0, \tilde{\ell}_1) - a + b$
- Graded symplectomorphisms preserve $\tilde{\mu}$
- $\tilde{\mu}(\tilde{\ell}_1, \tilde{\ell}_0) = n - \tilde{\mu}(\tilde{\ell}_0, \tilde{\ell}_1)$ (mod N)
- If u is a strip



$$\mu(x,y,u) \equiv \tilde{\mu}(\widetilde{T_xL_0}, \widetilde{T_xL_1}) - \tilde{\mu}(\widetilde{T_yL_0}, \widetilde{T_yL_1}) \pmod N$$

(Thus the Floer coboundary has degree $+1$ $dy = \sum n(x,y)x$)

- If $\phi(L)$ is push off of L by Morse function, $\tilde{\mu}$ = morse index (mod N)

- applications: suppose L admits a morse function with only even index critical points.

  Then $H^1(L;\mathbb{Z})=0 \Rightarrow L$ orientable $\Rightarrow L$ is 2-gradable.

  Floer complex $CF^i(L,L)$ concentrated in degree 0, so no differential:

$$HF^i(L,L) \simeq CF^i(L,L) \simeq C^i_{Morse}(L) \simeq H^i(L;\mathbb{Z}/2)$$

(Assuming $HF^i(L,L)$ can be defined)

Here's another: **Thm** (Seidel) Any Lagrangian submanifold $L \subset \mathbb{CP}^n$

satisfies $H^1(L;\mathbb{Z}/(2n+2)) \neq 0$

Since $N_{\mathbb{CP}^n} = n+1$, we have Maslov cover for $N \mid 2n+2$.

Key idea: $\mathbb{CP}^n$ admits a Hamiltonian circle action
$t \in \mathbb{R}/\mathbb{Z} \mapsto \sigma(t) = \mathrm{diag}(e^{2\pi i t}, 1, \ldots, 1) \in U(n+1) \subset \mathrm{Symp}(\mathbb{CP}^n)$

Since $H^1(\mathbb{CP}^n, \mathbb{Z}) = 0$, this loop lifts to a path of graded symplectomorphisms. $\tilde{\sigma} : [0,1] \to \mathrm{Symp}^{gr}(\mathbb{CP}^n)$
$\sigma(0) = \mathrm{id}$

Look at how $\tilde{\sigma}$ acts at a fixed point on a Lagrangian
The induced loop has Maslov index 2,
so $\tilde{\sigma}(1)$ = shift of grading by 2 = $[-2]$
Assume L is N-gradable.
Thus any $\tilde{L}$ is Hamiltonian isotopic to $\tilde{L}[-2]$
By hamiltonian invariance

$$HF^i(\tilde{L},\tilde{L}) \simeq HF^i(\tilde{L},\tilde{L}[-2]) \simeq HF^{i-2}(\tilde{L},\tilde{L})$$

So $HF^*(\tilde{L},\tilde{L})$ must be 2-periodic if L is N-gradable.

Proof of theorem: For a contradiction, suppose L is a Lagrangian with $H^1(L;\mathbb{Z}/(2n+2)) = 0$.

- Then L is $(2n+2)$-gradable and $HF^*(\tilde{L},\tilde{L})$ is $\mathbb{Z}/(2n+2)$-graded
- Implies $(2n+2) \mid N_L$ so $N_L \geq 2n+2$

- Also $H^1(L;\mathbb{Q}) = 0$. Thus the monotonicity of $\mathbb{CP}^n$ implies the monotonicity of L.

[All of $H_2(M,L)/\text{torsion}$ comes from $H_2(M)$ up to integer multiple.]

Since L is monotone with $N_L \geq 2n+2 \geq 3$
Oh spectral sequence applies.
the E_1 page is

$$E_1^i = \begin{cases} H^i(L;\mathbb{Z}/2) & \text{if } 0 \leq i \leq n \\ 0 & \text{if } n+1 \leq i \leq 2n+1 \end{cases}$$

The next differential in the Oh SS would have degree $N_L \geq 2n+2$ hence vanishes for degree reasons.

Thus $HF^i(\tilde{L},\tilde{L}) \simeq H^i(L;\mathbb{Z}/2)$ with grading reduced modulo $2n+2$
But this is not 2 periodic! Contradiction.