Introduction to the Fukaya Category.

The Floer homology groups we have been considering fit into a larger algebraic structure called the Fukaya category.

Assume $(M, \omega = d\theta)$ is exact, all Lagrangians below exact.
We take coefficients in $\mathbb{Z}/2$.

$\mathcal{F}(M,\omega)$: Objects = Lagrangian submanifolds (possibly with extra decorations)

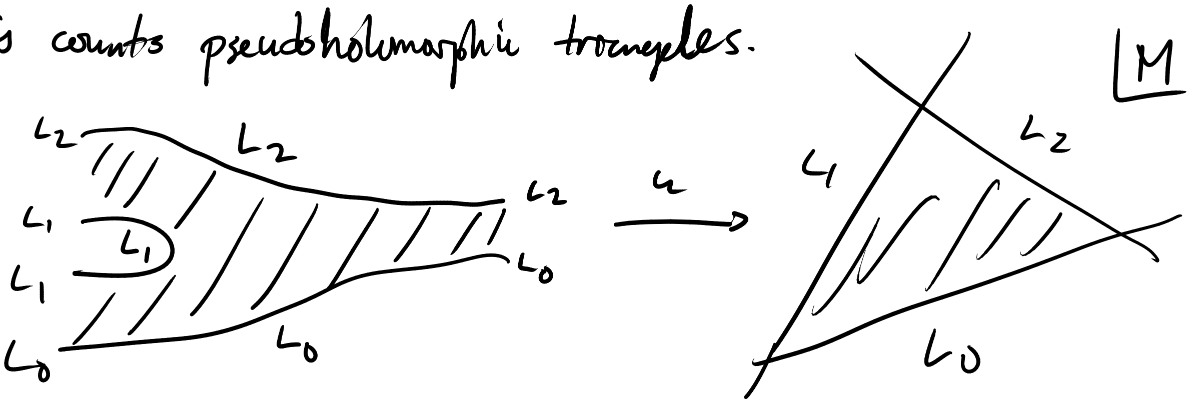
Morphism Spaces $\quad \mathrm{Hom}(L_0, L_1) = HF(L_0, L_1)$

Need to define composition of morphisms

$$\begin{tikzcd} L_0 \arrow[r, "x"] \arrow[rr, bend left, "\mu^2(x,y)"] & L_1 \arrow[r, "y"] & L_2 \end{tikzcd}$$

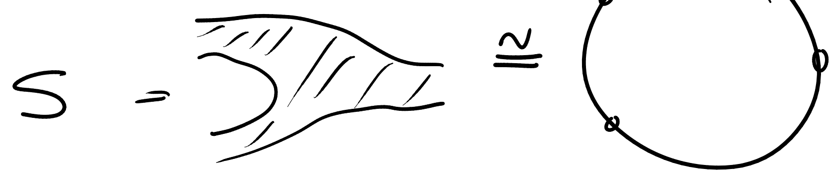
$$\mu^2 : HF(L_1, L_2) \otimes HF(L_0, L_1) \to HF(L_0, L_2)$$

This counts pseudoholomorphic triangles.



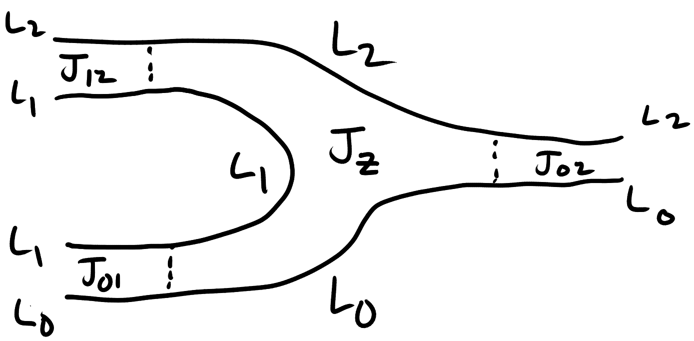
By a "triangle", we mean a map from a disk with three strip-like ends, as depicted above.

This is conformally equivalent to a disk with three punctures on the boundary



To define the moduli space: Assume L_0, L_1, L_2 are pairwise transverse. Pick almost complex structures J_{01}, J_{12}, J_{02} such that $HF(L_i, L_j, J_{ij})$ is well defined.
That is to say, J_{ij} is a complex structure such that moduli spaces of strips with boundary on L_i and L_j are transverse.
(This choice is necessary because μ^2 can only be defined for a particular chain-level model of $HF(L_i, L_j)$)

Now we pick a family of almost complex structures J_z, $z \in S$
(That is, the target space complex structure depends on the point in the domain)
This family should agree with J_{ij} over the end corresponding to (L_i, L_j)



Now let $x_{ij} \in L_i \cap L_j$ basis element of $CF(L_i, L_j)$

Define $\mathcal{M}(x_{01}, x_{12}, x_{02}) = \{ (j, J_z)$-holomorphic maps $u : S \to M$ satisfying boundary conditions asymptotic to $x_{01}, x_{12}, x_{02} \}$

Look at 0-dimensional components

$$\mu^2(x_{12}, x_{01}) = \sum_{x_{02}} \# \mathcal{M}(x_{01}, x_{12}, x_{02})^0 x_{02}$$

This actually defines a map

$$\mu^2 : CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$$

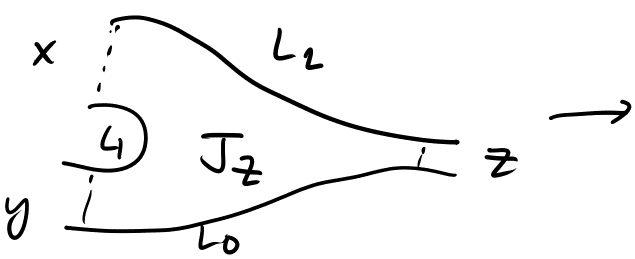
To see that it is well-defined on homology we need to prove

$$\partial(J_{02})\, \mu^2(x, y) = \mu^2(x, \partial(J_{01})\, y) + \mu^2(\partial(J_{12})x, y)$$

" μ^2 is a chain map" or " ∂ is a derivation of μ^2 "

(Note we are ignoring signs by working mod 2.)

To prove this, use 1-dimensional components of $\mathcal{M}(y, x, z)$

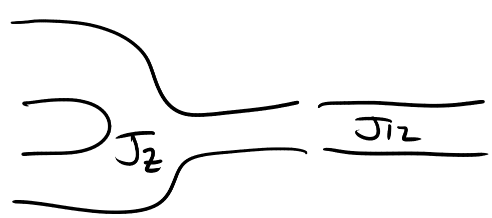


$\mu^2(\partial x, y)$

$+$

$\mu^2(x, \partial y)$

$+$



$\partial \mu^2(x, y)$

Notice how the complex structure on the strip depends on which end the breaking occurs at.

The desired identity holds because the boundary of the one-dimensional moduli space consists of an even # of points.

In order for these μ^2 compositions to define a category they must satisfy an associative law

$$L_0 \xrightarrow{x_{01}} L_1 \xrightarrow{x_{12}} L_2 \xrightarrow{x_{23}} L_3$$

$$\mu^2(x_{23}, \mu^2(x_{12}, x_{01})) = \mu^2(\mu^2(x_{23}, x_{12}), x_{01})$$

as maps $\quad HF(L_2, L_3) \otimes HF(L_1, L_2) \otimes HF(L_0, L_1) \to HF(L_0, L_3)$

The way to do this is to show that the difference between the two sides is a null homotopic map on the chain level.

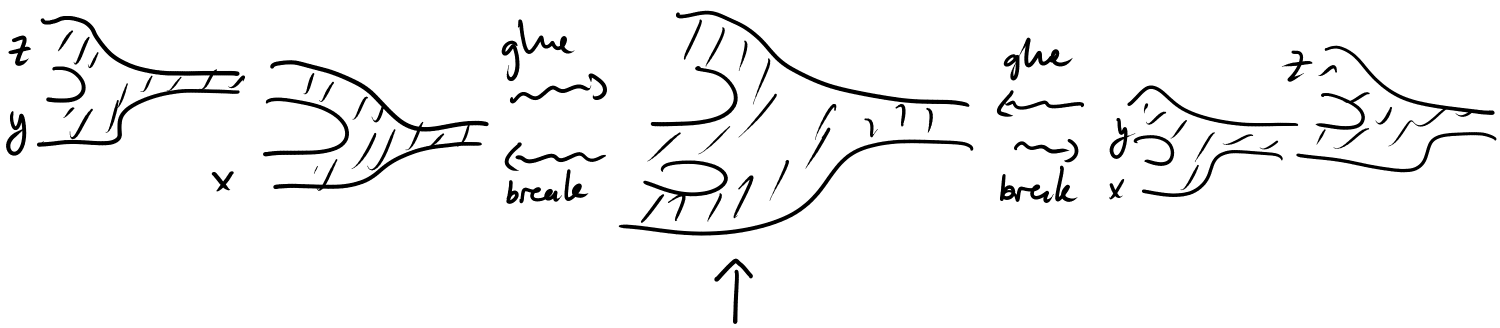
We need an operator, say $P(x,y,z)$, such that

$$\partial P(x,y,z) + P(\partial x,y,z) + P(x,\partial y,z) + P(x,y,\partial z)$$

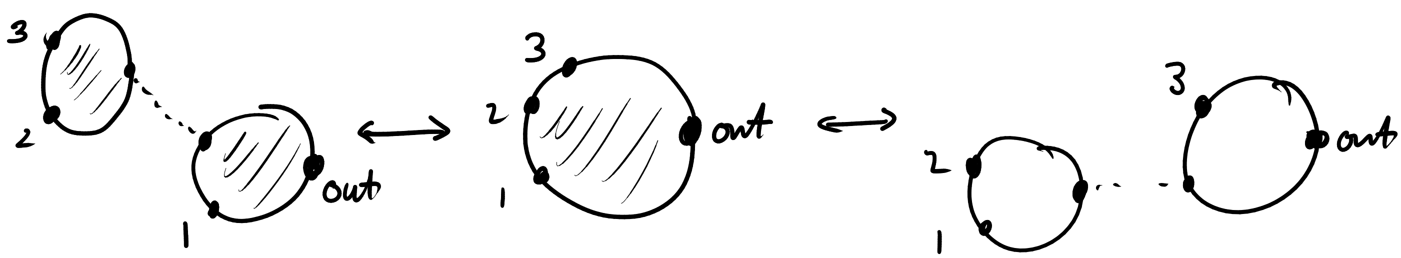
$$= \mu^2(x,\mu^2(y,z)) - \mu^2(\mu^2(x,y),z)$$

This operator P is actually called μ^3.

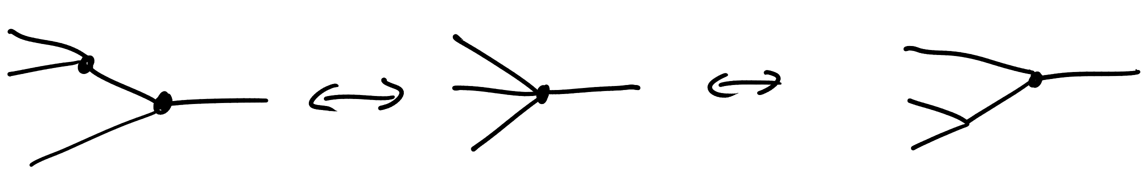
Think about composing two μ^2's



This Riemann surface is equivalent to a disk with 4 boundary punctures. It has one real modulus.



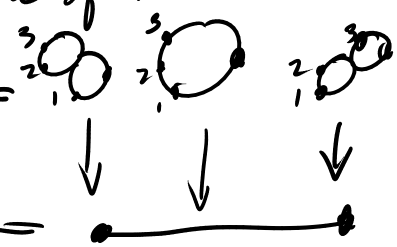
There is a related picture in terms of graphs (dual complexes)



The moduli space of disks with 4 boundary punctures is an interval

$\mathcal{S}^4 =$

$\mathcal{R}^4 =$



We let $\mathcal{S}^4$ denote the universal family. S_r the fiber over $r \in \mathcal{R}^4$

We allow the target almost complex structure to depend on $z \in \mathcal{S}^4$ (depends on moduli r and point on S_r) as well as on the Lagrangians involved L_0, L_1, L_2, L_3

The choice of J on $\mathcal{S}^4$ must be compatible with previous choices.

1) Over strip-like ends of $S_r \subseteq \mathcal{S}^4$, say one with labels L_i, L_j J must coincide with J_{ij}.

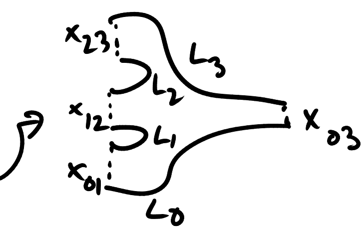
2) Over boundary points of $\mathcal{R}^4$, J must coincide with the J's previously chosen in the definition of μ^2

Reason: 1) gaurantees that when strips break off, the are holomorphic for the correct J_{ij} that defines the differential.

2) gaurantees that our μ^3 will actually be a homotopy relating the μ^2 operations defined earlier.

μ^3 is defined by looking at

$$\mathcal{M} = \left\{ (r \in \mathbb{R}^4, u: S_r \to M) \;\middle|\; \begin{array}{l} u \text{ is pseudoholomorphic} \\ u \text{ satisfies boundary} \\ \text{and asymptotic conds} \end{array} \right\}$$



I.e.

$$\mu^3(x_{23}, x_{12}, x_{01}) = \sum_{x_{03}} \#\left(\mathcal{M}(x_{23}, x_{12}, x_{01}, x_{03})^0\right) x_{03}$$

count 0-dimensional components

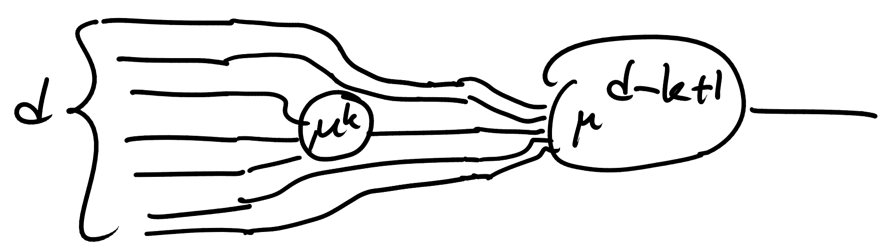
To prove homotopy property, use 1-dimensional components and cobordism argument.

There are higher operations μ^d, $d \geq 1$ (note $\mu^1 = \partial$)
Defined by extending the same idea to $(d+1)$ punctured disks.

These are "higher homotopies" in general, the equation looks like

$$\partial\mu^d(x_1, \dots, x_d) + \mu^d(\partial x_1, \dots, x_d) + \dots + \mu^d(x_1, \dots, \partial x_d)$$

$= \sum$



The terms in the sum are indexed by trees with "two internal vertices" (valence > 1)

The operations $\{\mu^d\}_{d\geq 1}$ make $\mathcal{F}(M,\omega)$, with

$\mathrm{Hom}(L_0,L_1) = CF(L_0,L_1)$ into an A_∞-category.

<u>Moral</u>: The A_∞ relations naturally arise from the moduli spaces of disks we use to define the category.