Fredholm differential topology

symbol invertible (compact case) / Need to understand boundary/asymptotic conditions otherwise

A general pattern is that $\boxed{\text{Elliptic}}$ PDE lead to **Fredholm operators**. — functional analytic notion.

<u>Def</u> Let X and Y be Banach spaces. A continuous linear map $D: X \to Y$ is called <u>Fredholm</u> if

(i) $\ker D$ is finite dimensional and
(ii) $\operatorname{Im} D$ is a closed subspace of finite codimension

<u>Rem.</u> The condition that $\operatorname{Im} D$ be closed is actually redundant.

<u>Def</u> The <u>index</u> of the Fredholm operator D is
$$\operatorname{ind}(D) = \dim(\ker D) - \dim(\operatorname{coker} D)$$

- The index is remarkably stable under perturbation:

<u>Theorem</u>: Let $D: X \to Y$ be a Fredholm operator.

(a) For any compact operator $K: X \to Y$, $D+K$ is Fredholm and $\operatorname{ind}(D+K) = \operatorname{ind}(D)$

(b) $\exists\, \varepsilon > 0$ such that if $\|P\| < \varepsilon$ (operator norm) then $D+P$ is Fredholm and $\operatorname{ind}(D+P) = \operatorname{ind}(D)$

Assertion (b) means that the set of Fredholm operators is open in the operator norm topology, and the index is a locally constant function on this set.

(The operator K is compact if $K(\text{unit ball})$ has compact closure)

An equivalent definition of Fredholm operators is that they are invertible up to compact operators

For a Fredholm operator $D : X \to Y$, there is a bounded linear operator $P : Y \to X$ such that

$$PD = Id_X + K_X \qquad DP = Id_Y + K_Y$$

where K_X, K_Y are compact operators.

In fact, we may arrange that K_X and K_Y have finite-dimensional images.

We will need to go beyond the linear case so suppose

$f : X \to Y$ is continuous map of Banach spaces

For $x \in X$, Derivative $df_x : X \to Y$ is the bounded linear map such that

$$\| f(x+h) - f(x) - df_x h \|_Y = o(\|h\|_X)$$

Provided such a map exists of course.

f is <u>continuously differentiable</u> if df_x exists, and the map $\left\{ \begin{array}{l} X \to L(X,Y) \\ x \mapsto df_x \end{array} \right\}$ is continuous, where $L(X,Y)$ has operator norm.

C^l maps are defined inductively by requiring df to be C^{l-1}.

A differentiable map $f: X \to Y$ is called <u>Fredholm</u> if $df_x : X \to Y$ is a (linear) Fredholm operator for all $x \in X$.

If any one out there is afraid of infinite dimensions, it is important to realize that Fredholm maps have an excellent differential topology, which closely tracks the finite dimensional theory.

<u>Def</u> $y \in Y$ is a <u>regular value</u> of f if

df_x is surjective for all $x \in f^{-1}(y)$

<u>Thm</u> (Sard-Smale) Let $f: X \to Y$ be smooth (C^∞)

Then the set of regular values of f is the countable intersection of open, dense sets ("second category in the sense of Baire")

<u>Thm</u> (Implicit function theorem)

let $f: X \to Y$ be smooth, and let $y \in Y$ be a regular value.
Then $f^{-1}(y)$ is a smooth Banach submanifold of X with an isomorphism

$$T_x(f^{-1}(y)) \cong \ker(df_x)$$

<u>Cor</u> If f is Fredholm, $f^{-1}(y)$ is a smooth manifold of finite dimension and

$$\dim f^{-1}(y) = \operatorname{ind}(df_x)$$

This implicit function theorem is how we (may) get smooth finite dimensional moduli spaces from Fredholm maps (which in turn arise from elliptic PDE)

Recall from a previous lecture our description of the linearization of the pseudoholomorphic map equation.

$$u : S \longrightarrow M \qquad \bar\partial_J u = \tfrac{1}{2}(du + J \circ du \circ j) \in \Omega^{0,1}(S, u^*TM)$$

S: Riemann surface $\quad$ M: symplectic manifold.

Space of maps $\quad \mathcal{B} = \mathrm{Map}(S, M)$

Tangent space $\quad T_u\mathcal{B} = \Gamma(S, u^*TM)$

Value of $\bar\partial_J u$ is in $\mathcal{E}_u = \Omega^{0,1}(S, u^*TM)$

There is a bundle

$$\begin{tikzcd} \mathcal{E}_u \arrow[d] & \subset & \mathcal{E} \arrow[d] \\ u & \in & \mathcal{B} \arrow[u, bend right, "\bar\partial_J"'] \end{tikzcd}$$

$\bar\partial_J$ is a section.

In a local chart on $\mathcal{B}$, viz. $\mathcal{U} \subset \Gamma(S, u^*TM)$ and a local trivialization of $\mathcal{E}$

$$\mathcal{E} \simeq \mathcal{U} \times \Omega^{0,1}(S, u^*TM)$$

(we constructed this using the exponential map)

$\bar\partial_J$ becomes a map $\quad \bar\partial_J : \underset{\Gamma(S, u^*TM)}{\mathcal{U} \cap} \longrightarrow \Omega^{0,1}(S, u^*TM)$

whose linearization at the center of the chart u is

$$D_u \bar{\partial}_J = \nabla_s + J \nabla_t + (\text{lower order}),$$ a Cauchy-Riemann type operator.

Now, <u>none</u> of the preceding functional analysis applies because we are considering C^∞ objects, and the topology of C^∞ is not induced by a norm, so we do not have Banach spaces.

The game now becomes to find norms where things work. The conventional solution is to use Sobolev spaces L^p_k

So replace $\mathcal{B} = \text{Maps}(S, M)$ with $\mathcal{B}^p_k = L^p_k\text{-Maps}(S, M)$

<u>Note</u>: if $kp > \dim S = 2$, such maps are continuous, so this makes topological sense.

Since $\bar{\partial}_J$ involves one derivative, it can only be continuous and Fredholm if we put the L^p_{k-1} topology on the target:

So replace $\Omega^{0,1}(S, u^*TM)$ with the L^p_{k-1} completion.

Then (under suitable boundary and asymptotic conditions)

$\bar{\partial}_J$ is a Fredholm section, locally modeled on a map between the Banach spaces

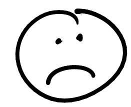
$$\begin{array}{c} \mathcal{U} \longrightarrow L^p_{k-1}\text{-}\Omega^{0,1}(S, u^*TM) \\ \cap \\ L^p_k(S, u^*TM) \end{array}$$

In the happy case that 0 is a regular value of this map, then

the set of J-holomorphic maps

$$\bar{\partial}_J^{-1}(0) = \{ u : S \to M \mid \bar{\partial}_J u = 0 \}$$

is a smooth manifold of dimension equal to the Fredholm index of $D_u \bar{\partial}_J$, as a consequence of the abstract "Fredholm differential topology".

But what if zero isn't a regular value? 

1) just change the problem (e.g. perturb J, add further inhomogeneous terms to the equation, requires some geometric cleverness.)

2) Use more sophisticated "Fredholm differential topology" (Kuranishi structure, Virtual neighborhood, Polyfolds, ...)