# Outline of Floer's construction

Last time, we approached Morse theory ~~for~~ the Action functional

$(M,\omega)$ symplectic $\qquad L_0, L_1 \subset M$ Lagrangian.
$J$ compatible a.c.s., $g(\cdot,\cdot)=\omega(\cdot, J\cdot)$ metric.

$$\Omega(L_0,L_1) = \left\{ \gamma : [0,1] \to M \;\middle|\; \begin{array}{l} \gamma(0) \in L_0 \\ \gamma(1) \in L_1 \end{array} \right\}$$

$$\mathcal{A} : \Omega(L_0,L_1) \longrightarrow \mathbb{R} \quad \text{(possibly multivalued)}$$

$$T_\gamma \Omega(L_0,L_1) = \left\{ X \in \Gamma(\gamma^* TM) \;\middle|\; \begin{array}{l} X(0) \in T_{\gamma(0)} L_0 \\ X(1) \in T_{\gamma(1)} L_1 \end{array} \right\} \quad \text{"smooth tangent space"}$$

$$d\mathcal{A}_\gamma : T_\gamma \Omega(L_0,L_1) \longrightarrow \mathbb{R}$$

$$d\mathcal{A}_\gamma(X) = \int_0^1 \omega(X, \dot\gamma)\,dt$$

Gradient $\nabla \mathcal{A}_\gamma = -J\dot\gamma$

Critical points: $\dot\gamma \equiv 0 \iff \gamma \equiv p \in L_0 \cap L_1$

~~Hessian~~ $\quad \mathrm{Hess}_\gamma \mathcal{A}(X,Y) = \displaystyle\int_0^1 \omega(X, \dot Y)\,dt \qquad X,Y : [0,1] \to T_pM$

As a operator $\quad -J\dfrac{d}{dt} :$ "$T_\gamma \Omega(L_0,L_1) \to T_\gamma \Omega(L_0,L_1)$"

Critical point nondegenerate $\iff$ $L_0$ and $L_1$ intersect transversely at $p$.

Index is always infinite.

There are some serious issues here.

(1) We want to study gradient flow of $\mathcal{A}$ on $\Omega(L_0, L_1)$.
But $\nabla \mathcal{A}_\gamma$ is not even tangent to $\Omega(L_0, L_1)$!
i.e. $J\dot{\gamma}(0)$ need not be tangent to $L_0$, sim. for $L_1$

How could things have gone so horribly wrong?
In infinite dimensions, the existence of the gradient is only gauranteed by the Riesz Representation theorem, which only applies to Hilbert spaces.

Since the inner product is an $L^2$ inner product we only know that $\langle X, Y \rangle = \int_0^1 g_{\gamma(t)}(X(t), Y(t))\,dt$

$\circledast$ $\nabla \mathcal{A}_\gamma$ lies in the $L^2$-completion of $T_\gamma \Omega(L_0, L_1)$

<u>Homework</u>: Prove $\circledast$, i.e. find an $L^2$ Cauchy sequence $X_n \in T_\gamma \Omega(L_0, L_1)$ that converges in $L^2$ to $\nabla \mathcal{A}_\gamma = -J\dot{\gamma}$

We could try to use a "stronger" metric, eg. $\int g(X,Y) + g(\nabla X, \nabla Y)$ but this does not lead to Floer homology.

(2) The Morse index is infinite. This is not a "technical problem", as it fundametally affects any attempt to link the critical points of the action functional to the topology of the Path space.

Floer's approach is to side step (1) by considering a PDE rather than an ODE.

For (2), Floer defines a "relative index" depending on a path between two critical points. This turns out to be intimately linked to the index theory of the PDE.

à la Atiyah-Singer et al.

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Floer's equation (J-holomorphic strip version)

Pretend we had a reasonable gradient flow

$$\gamma_S : \mathbb{R}_S \longrightarrow \Omega(L_0, L_1)$$

$$\frac{d}{ds}\gamma_s = \nabla A_{\gamma_s} = -J\dot{\gamma}_s$$

Thus $\gamma_s$ is a path of paths, which we may also regard as a parametrized surface $u(s,t)$

$$u(s,t) : \underset{s}{\underset{\Psi}{\mathbb{R}}} \times \underset{t}{\underset{\Psi}{[0,1]}} \longrightarrow M \qquad u(s,\cdot) = \gamma_s$$

As $\gamma_s \in \Omega(L_0, L_1)$ we have

$$(\forall s \in \mathbb{R})\left( u(s,0) \in L_0 \text{ and } u(s,1) \in L_1 \right)$$

Gradient flow becomes PDE

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$$

$$\iff \quad du\left(\frac{\partial}{\partial s}\right) + J_{u(s,t)}\, du\left(\frac{\partial}{\partial t}\right) = 0 \quad \in T_{u(s,t)}M$$

$\Longleftrightarrow$ Think of $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ as sections of
$$\begin{array}{c} u^*TM \\ \downarrow \\ \mathbb{R}\times[0,1] \end{array}$$

and $u^*J$ is an acs on this bundle

Then
$$\frac{\partial u}{\partial s} + u^*J\frac{\partial u}{\partial s} = 0 \quad \text{as section of } u^*TM$$

$\Longleftrightarrow$ give $\mathbb{R}\times[0,1]$ a complex structure so that $s+it$ is a holomorphic function.

i.e. $j \in \mathrm{End}\left(T(\mathbb{R}\times[0,1])\right)$

such that $j\left(\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial t} \qquad j\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial s}$

Then $du \circ j = J \circ du$ i.e. $du$ $\mathbb{C}$-linear

i.e. $u$ is $J$-holomorphic

To see the last equivalence, look at

$$du\circ j\left(\frac{\partial}{\partial s}\right) = du\left(\frac{\partial}{\partial t}\right) = \frac{\partial u}{\partial t} \qquad J\circ du\left(\frac{\partial}{\partial s}\right) = J\cdot du\left(\frac{\partial}{\partial s}\right) = J\frac{\partial u}{\partial s}$$

$$du\circ j\left(\frac{\partial}{\partial t}\right) = du\left(-\frac{\partial}{\partial s}\right) = -\frac{\partial u}{\partial s} \qquad J\circ du\left(\frac{\partial}{\partial t}\right) = J\,du\left(\frac{\partial}{\partial t}\right) = J\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial t} = J\frac{\partial u}{\partial s} \Longleftrightarrow J\frac{\partial u}{\partial t} = J^2\frac{\partial u}{\partial s} \Longleftrightarrow J\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial s}$$

so the two component equations are equivalent, and equivalent to
$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$$

For this reason, solutions of the equation are called "pseudo-holomorphic strips".

Recalling the desired analogy to Morse theory, we are going to look for strips that join "critical points" i.e. constant paths.

So let $\gamma_+ \equiv p_+ \in L_0 \cap L_1$ be intersection points
$\gamma_- \equiv p_- \in L_0 \cap L_1$

We will require
asymptotic conditions
$$\lim_{s\to\infty} u(s,t) = \gamma_+(t) \equiv p_+ \in L_0 \cap L_1$$
$$\lim_{s\to-\infty} u(s,t) = \gamma_-(t) \equiv p_- \in L_0 \cap L_1$$

Define the "Moduli space"

$$\mathcal{M}(p_-,p_+) = \Big\{ u : \mathbb{R}\times[0,1] \to M \;\Big|$$

(J-hol strip) $\quad \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$

(Lagrangian boundary condition) $\quad (\forall s\in\mathbb{R})\,\big(u(s,0)\in L_0 \text{ and } u(s,1)\in L_1\big)$

(asymptotic condition) $\quad \lim_{s\to\pm\infty} u(s,t) = p_\pm \Big\}$

This is just a set right now, but we want to make it into a smooth manifold, (locally) of finite dimension.

Then we will use $\mathcal{M}(p_-,p_+)$ in a way strictly analogous to the spaces of gradient trajectories in Morse theory:

- There is an $\mathbb{R}$-action, which is smooth, free, and proper, so the quotient is a manifold.

$$\mathcal{M}(p_-, p_+)/\mathbb{R}$$

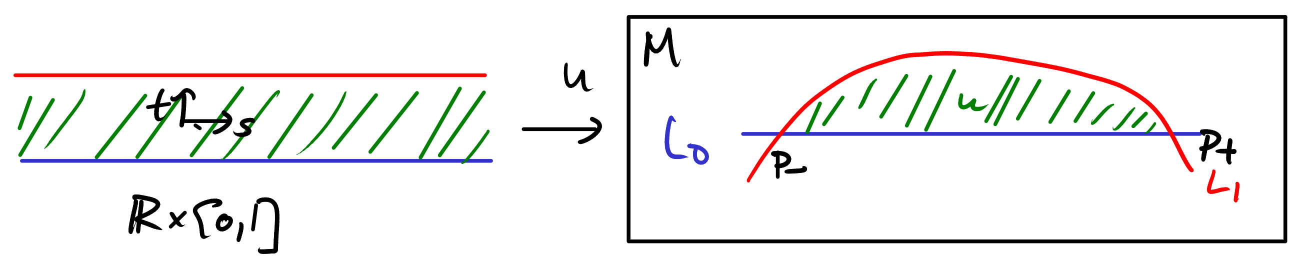
- There is a "Compactness modulo breaking" theorem, showing that the zero dimensional components of $\mathcal{M}(p_-, p_+)/\mathbb{R}$ are compact, so we can count them to define a differential $d$

- However, we find that the 1-dimensional components are not necessarily compactified by products of 0-dimensional components. This is a serious issue that affects how generally Floer homology can be defined.

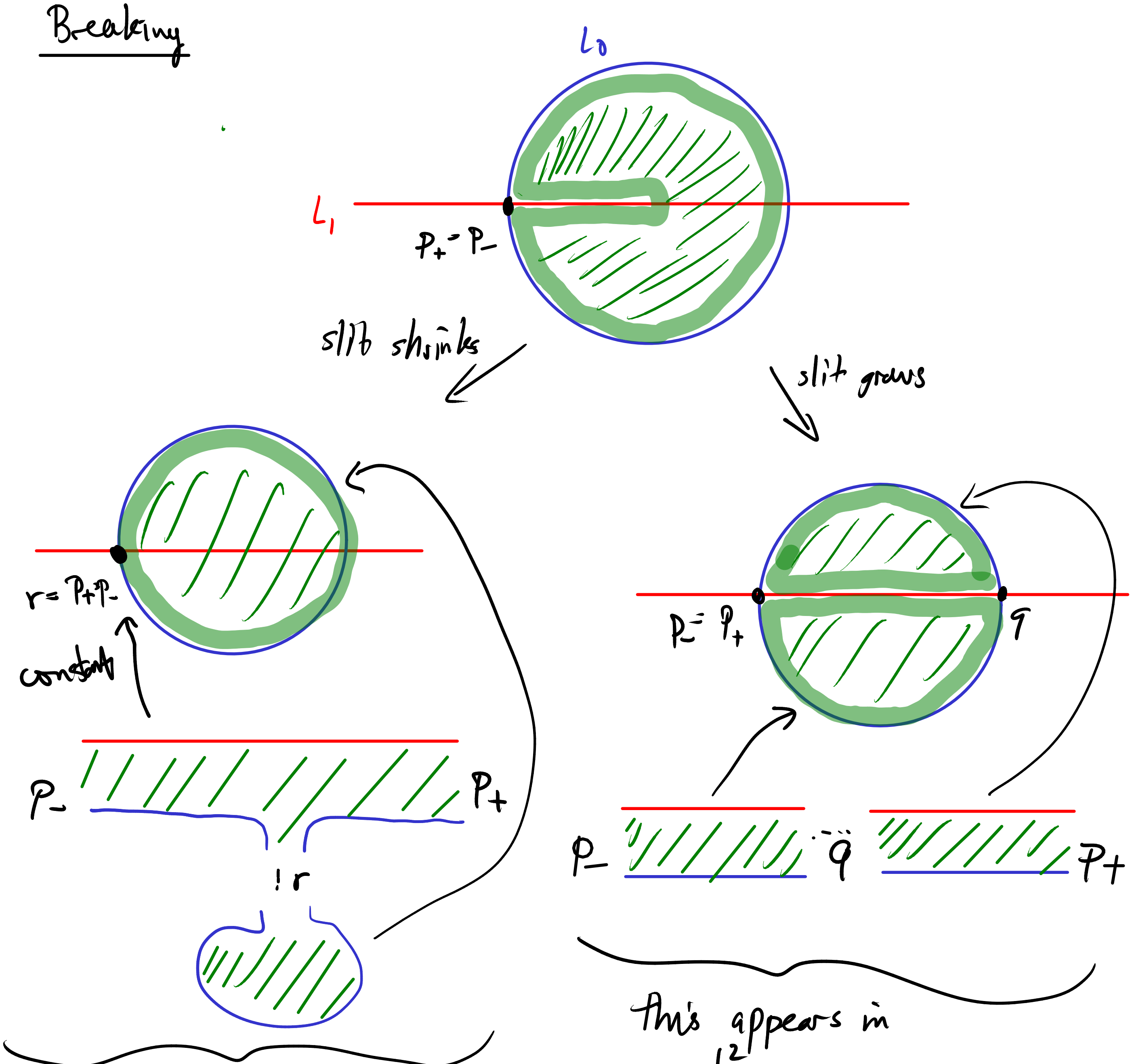
- But, once that issue is dealt with somehow, we obtain the relation $d \circ d = 0$ by considering boundaries of 1-dimensional components of $\mathcal{M}(p_-, p_+)/\mathbb{R}$

<u>Pictures:</u>

<u>Strip:</u>



# Breaking




This appears in $d^2$

This is a "bad degeneration"
AKA obstruction
AKA $\mu^0$ in <u>curved</u> $A_\infty$-structure.