Elliptic differential operators and the Fredholm property: The analytic context of Floer homology

Let recall our equation for a map $u : \underset{s}{\mathbb{R}} \times \underset{t}{[0,1]} \to M$, where $\Theta = \mathbb{R}\times[0,1]$

$$\frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0 \quad \text{as section of} \quad u^* TM \to \Theta$$

This is a first-order nonlinear PDE

why nonlinear? (1) Target M is not even a vector space.

(2) even locally, in coordinates $(x^a)_{a=1,\dots,2n}$ on M

$u(s,t) = (u^a(s,t))_{a=1}^{2n}$ equation becomes

$$\frac{\partial u^a}{\partial s} + \underbrace{J^a_b(u)}\frac{\partial u^b}{\partial t} = 0$$

This coefficient matrix depends on unknown function u

We can't avoid this unless J is integrable.

We will still apply the analysis of linear PDE by studying the <u>linearization</u> of the equation.

This means, essentially that need to differentiate the expression

$$\frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t}$$

in the direction of sm vector field

$$X \in \Gamma(\Theta, u^*TM) = T_u \mathrm{Map}(\Theta, M)$$

To do this, we choose a connection ∇ on M, the Levi Civita connection of $g = \omega(\cdot, J\cdot)$.

Then a local chart on $\mathrm{Map}(\Theta, M)$ near u is given by

$$X \in \mathcal{V} \mapsto \tilde{u}_X(s,t) = \exp_{u(s,t)}(X(s,t)) \in \mathrm{Map}(\Theta, M)$$

$$\mathcal{V} \subset \Gamma(\Theta, u^*TM) \qquad \exp \text{ w.r.t. } g, \nabla$$

The value of $\frac{\partial \tilde{u}_X}{\partial s} + J\frac{\partial \tilde{u}_X}{\partial t}$ lives in $\Gamma(\Theta, \tilde{u}_X^{\,*}TM)$

which is a different space for different $X \in \mathcal{V}$

The derivation of the exponential map $\exp : TM \to M$

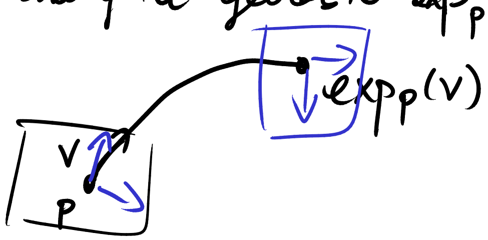
$$D\exp : T(TM) \to TM$$

$$T(TM) \supset T_{vert}(TM)$$

$$\widetilde{\exp} : T_{vert}(TM) \ni (p, v, \Delta v) \longmapsto (\exp_p(v), \widetilde{\Delta v})$$

$$p \in M, \quad v \in T_pM, \quad \Delta v \in T_vT_pM = T_pM, \quad \widetilde{\Delta v} \in T_{\exp_p(v)}M$$

identifies T_pM with $T_{\exp_p(v)}M$ along the geodesic $\exp_p(tv)$



We use these identifications to trivialize

$$\Gamma(\Theta, \tilde{u}_X^{\,*}TM) \simeq \Gamma(\Theta, u^*TM)$$

with these identifications, the expression $\frac{\partial \tilde{u}_x}{\partial s} + J(\tilde{u}_x)\frac{\partial \tilde{u}_x}{\partial t}$ defines a map

$$\begin{array}{c} x \in \mathcal{V} \longrightarrow \Gamma(\Theta, u^*TM) \\ \cap \\ \Gamma(\Theta, u^*TM) \end{array}$$

The linearization D_u is the derivative of this at the origin.

$$D_u : \Gamma(\Theta, u^*TM) \to \Gamma(\Theta, u^*TM)$$

<u>Prop</u>: $D_u \xi = \nabla_s \xi + J\nabla_t \xi + (\nabla_\xi J)\cdot\frac{\partial u}{\partial t}$

<u>Comments</u>: ∇_s is $(\underset{\substack{\uparrow \\ \text{fixed}}}{u}^*\nabla)_{\frac{\partial}{\partial s}}$ $\qquad$ ∇_t is $(\underset{\substack{\uparrow \\ \text{fixed}}}{u}^*\nabla)_{\frac{\partial}{\partial t}}$

J is $\underset{\substack{\uparrow \\ \text{fixed}}}{u}^*J$ $\qquad$ $\frac{\partial u}{\partial t}$ is a fixed vector field

$\nabla_\xi J$ is a matrix which depends linearly on ξ.

<u>Homework</u>: prove this. If too difficult, just prove

$$D_u \xi = \nabla_s \xi + \nabla_t \xi + B\xi$$

$\leftarrow$ B some zeroth order operator $\in \mathrm{End}(u^*TM)$

(This less precise formulation doesn't depend on connection ∇)

$$\nabla_s \xi \longmapsto \nabla_s \xi + \underset{\substack{\cap \\ \mathrm{End}(u^*TM)}}{A}\xi$$

Now, in a fixed trivialization of (u^*TM, u^*J)
$$\downarrow$$
$$\Theta$$

We can think of sections as just $2n$-tuples of functions

$$D_u \xi = \nabla_s \xi + J\nabla_t \xi + (\nabla_\xi J)\frac{\partial u}{\partial t}$$

$$= \underbrace{\frac{\partial \xi}{\partial s} + J\frac{\partial \xi}{\partial t}}_{\text{This is a standard Cauchy-Riemann operator}} + \underbrace{\left(\Gamma_s \cdot \xi + \Gamma_t \xi + \nabla_\xi J \cdot \frac{\partial u}{\partial t}\right)}_{B\cdot\xi}$$

This is a standard Cauchy-Riemann operator in the complex bundle (u^*TM, J) over the Riemann surface Θ (same thing from complex geometry)

$B \cdot \xi$
$B \in \mathrm{End}_{\mathbb{R}}(TM)$

Overall, D_u is an "$\mathbb{R}$-linear Cauchy-Riemann type operator"
In particular D_u is a <u>Elliptic Differential operator</u>.

To explain this consider a general local expression of a differential operator. Between bundles

$$\begin{matrix} E & & F \\ & \searrow \;\; \swarrow & \\ & M & \end{matrix}$$

$-$arbitrary base manifold.

$$D : \Gamma(M,E) \to \Gamma(M,F)$$

$$Du = \sum_{|\alpha| \le m} A^\alpha(x) \frac{\partial^{|\alpha|} u}{\partial x^\alpha} \qquad m = \text{order of } D$$

$\alpha = (\alpha_1, \dots, \alpha_n)$ multiindex

$A^\alpha(x) : E_x \to F_x$
w.r.t. local trivializations.

To get the symbol of D, replace $\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ by $i^{|\alpha|} p^\alpha$

where $p_1, \dots, p_n$ are coordinates dual to $x^1, \dots, x^n$

(really these are coordinates on the cotangent bundle T^*M)

$$\sigma(D) = \sum_{|\alpha| \leq m} A^\alpha(x) p^\alpha \qquad \text{"Total Symbol"}$$

The total symbol is not invariantly defined, but its leading order part

$$\sigma_m(D) = \sum_{|\alpha|=m} A^\alpha(x) p^\alpha$$ does have invariant meaning

$$\sigma_m(D) \in \Gamma\left(\mathrm{Sym}^m TM \otimes \mathrm{Hom}(E,F)\right)$$

Or: $\sigma_m(D)$ is a function on T^*M with values in $\pi^* \mathrm{Hom}(E,F)$ $\quad (\pi: T^*M \to M)$

which is homogeneous of degree m on the fibers of T^*M

__Def__ D is elliptic if $\sigma_m(D)(\alpha): E_x \to F_x$ is invertible for any $x \in M$ and nonzero covector $0 \neq \alpha \in T_x^*M$.

__Canonical example__ $\quad E = F = \underline{\mathbb{R}}$ trivial line bundle

$$\Delta : C^\infty(M) \to C^\infty(M) \qquad \text{Laplacian}$$

$$\Delta u = \frac{1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial u}{\partial x^k}\right) = \sum_{i,j} g^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \left(\begin{matrix}\leq 1\text{st} \\ \text{order}\end{matrix}\right)$$

factor of $i^m = i^2 = -1$

$$\sigma_2(\Delta) = -\sum g^{ij} p_i p_j$$

$$\underset{\alpha \in T_x^* M}{\sigma_2(\Delta)(\alpha)} = -\|\alpha\|^2$$

computed using Riemannian metric on covectors.

certainly invertible for nonzero α.
So Δ is elliptic.

<u>Homework</u> show D_u is elliptic according to the above definition.

Analytic meaning of the symbol: Take $M = \mathbb{R}^n$

Fourier transform $\quad \hat{u}(p) = (2\pi)^{-n/2} \int e^{-i\langle x,p\rangle} u(x)\,dx$

inverse transform $\quad u(x) = (2\pi)^{-n/2} \int e^{i\langle x,p\rangle} \hat{u}(p)\,dp$

$$\widehat{\frac{\partial u}{\partial x^j}} = i p_j \qquad \widehat{x^j \cdot u} = -i \frac{\partial \hat{u}}{\partial p_j}$$

Thus if $\displaystyle Du = \sum_{|\alpha| \le m} A^\alpha(x) \frac{\partial^{|\alpha|} u}{\partial x^\alpha}$

$$Du = (2\pi)^{-n/2} \int e^{i\langle x,p\rangle} \sigma(x,p)\, \hat{u}(p)\,dp$$

where $\displaystyle \sigma(x,p) = \sum_{|\alpha| \le m} A^\alpha(x)\, i^{|\alpha|} p^\alpha$ is the total symbol

Now $\sigma(x,p)$ is polynomial in $(p_1, \dots, p_n)$

If we allow more general functions in place of $\sigma(x,p)$, we obtain <u>Pseudodifferential operators.</u>

The most important functional analytic property of elliptic operators over a compact manifold M, is that they are <u>Fredholm</u>:

<u>Def</u> an operator $D: V_1 \to V_2$ between Banach spaces is Fredholm if
(i) $\ker D$ is finite dimensional
(ii) $\operatorname{Im} D$ is a closed subspace of finite codimension

(Thus $\operatorname{coker} D = V_2 / \operatorname{Im} D$ is finite dimensional)

<u>Def</u> The index of a Fredholm operator is
$$\operatorname{ind}(D) = \dim(\ker D) - \dim(\operatorname{coker} D)$$

<u>Thm</u> (Fundamental theorem of elliptic theory over compact M)
Let D be an elliptic operator of order m over a compact M
$$D: \Gamma(E) \to \Gamma(F)$$
For each $k \in \mathbb{Z}$, D extends to a bounded operator
$$D: L^2_k(E) \longrightarrow L^2_{k-m}(F)$$
This operator is Fredholm, with index independent of k

(Here, the Sobolev topology is defined by the norm
$$\|u\|_k^2 = \sum_{j=0}^{\infty} \int_X |\nabla\nabla\nabla\cdots\nabla u|^2 \, d\mathrm{vol}_X$$
defined using metrics and connections on M, E, F)