Dehn Twists:

An important and natural class of symplectic diffeomorphisms arises from the monodromy of families of algebraic varieties.

If $\begin{array}{c}\mathcal{X}\\ \pi\downarrow\\ B\end{array}$ is a projective family of varieties with generically smooth fibres, it becomes a symplectic fibration via pull back of the Fubini-Study form from the projective embedding.

There is some discriminant locus $\Delta \subset B$ over which the fibre is singular. Over $B^0 = B \setminus \Delta$, we have a symplectic fibration, By symplectic parallel transport (see lecture on Moser's trick), any path

$$\gamma : [0,1] \to B^0$$

we obtain a symplectomorphism $P_\gamma : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(1))$

In particular for a loop we get a symplective automorphism

$$\gamma : \mathbb{R}/\mathbb{Z} \to B^0 \rightsquigarrow P_\gamma \in \mathrm{Symp}\left(\pi^{-1}(\gamma(0))\right)$$

Homotopy of the loop γ changes P_γ by a Hamiltonian diffeo. Since $\mathrm{Ham}(\pi^{-1}(\gamma(0))) \subset \mathrm{Symp}(\pi^{-1}(\gamma(0)))_0$ Identity component.

The class of P_γ in $\pi_0(\mathrm{Symp}(\pi^{-1}(\gamma(0))))$ is determined by the Homotopy class of γ.

I.e. we have a map $\pi_1(B\setminus\Delta, b) \to \pi_0(\mathrm{Symp}(\pi^{-1}(b)))$

Q: What kind of symplectomorphism do we get this way?

Q: What is the difference between $\pi_0(\mathrm{Symp})$ and $\pi_0(\mathrm{Diff})$?

The automorphisms we get depends on the singularities of the family $\begin{array}{c}\mathcal{X}\\ \pi\downarrow\\ B\end{array}$.

Suppose the family contains a node: a singularity analytically equivalent to $\sum_{i=1}^{d+1} z_i^2 = 0$

Then the monodromy around such a singularity is a Generalized Dehn twist.

Local model: Consider $U = T^*S^n$
There is a local coordinate description

$$U \simeq \{(u, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |v| = 1, \ \langle u, v\rangle = 0\}$$

$$\omega_U = \sum_{j=1}^{n+1} du_j \wedge dv_j$$

$$\{z_1^2 + \cdots + z_n^2 = 1\} \simeq U$$

$$(z_1, \ldots, z_n) \longmapsto (-\mathrm{im}(z_i)|\mathrm{re}(z_i)|, \ \mathrm{re}(z_i)|\mathrm{re}(z_i)|^{-1})$$

There is a circle action on $T^*S^n \setminus S^n$ given by normalized geodesic flow for the round metric.

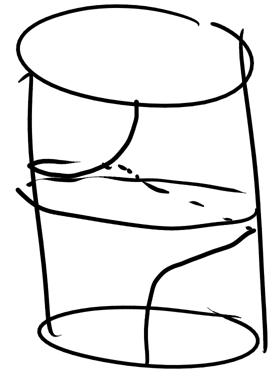
$$\sigma_t(u, v) = \left(\cos(2\pi t)u - \sin(2\pi t)v|u|, \ \cos(2\pi t)v + \sin(2\pi t)u|u|^{-1}\right)$$

$\sigma_{\frac{1}{2}}(u, v) = (-u, -v)$ extends our zero section

it is the antipodal map.

Let ψ be a function

Set $\tau(u,v) = \sigma_{\psi(|u|)}(u,v)$

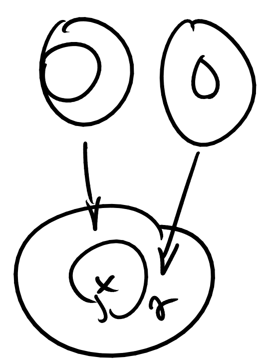


This is well defined on all of U.
$\tau = id$ outside of a compact neighborhood of the zero section, and it is antipodal map on zero section.

Any time we have a Lagrangian sphere l in M, we can embed a piece of the cotangent bundle into M via the Darboux-Weinstein theorem, and so we get a Dehn twist $\tau_l \in Symp(M)$

The class $[\tau_l] \in \pi_0(Symp(M))$ possibly depends on the parametrization of the sphere, but not the other choices.

For any nodal degeneration the monodromy around γ is τ_l for some Lagrangian sphere $l \hookrightarrow M$. (Known as the vanishing cycle.)



The action on $H_n(M)$ is given by the Picard Lefschetz Transformation:

$$(\tau_l)_*(c) = c - (-1)^{n(n-1)/2}(c\cdot[L])[L]$$

For a lagrangian, $[L]\cdot[L] = (-1)^{n(n-1)/2}\,\chi(L)$

For L sphere: $[L]\cdot[L] = \begin{cases} 0 & n \text{ odd} \\ 2 & n = 4k \\ -2 & n = 4k+2 \end{cases}$

Prop If n even, $(\tau_\ell)_*^2 = \mathrm{Id}$ on $H_n(M)$

If n odd, $(\tau_\ell)_*$ has infinite order provided $[L]$ is not a torsion class.

In fact Seidel has shown that in $n=2$, (real dim 4) τ_ℓ^2 is differentiably isotopic to the identity.
(τ_ℓ^2 is trivial in $\pi_0(\mathrm{Diff}(M))$)

But in general it is not symplectically isotopic to identity.

Homework: Determine the monodromy of the Hesse pencil

$$\{ y^2 = x(x-1)(x-\lambda) \}_{\lambda \in \mathbb{C}}$$

This is a pencil of elliptic curves in $\mathbb{CP}^2$, if we homogenize the equation. The general fiber is a torus. Singularities occur when $\lambda = 0$ or $\lambda = 1$.

Let's apply this to (M,ω) such that $2c_1(M,\omega) = 0$ with $n \geq 2$
Pick an ∞-fold Maslov covering. $\Lambda^\infty \to \Lambda \to M$.
Any L with $H^1(L) = 0$ is ∞-gradable, in particular spheres are always gradable.

Lemma The Dehn twist is gradable by a lift $\tilde{\tau}_\ell \circlearrowright \Lambda^\infty$ which is identity outside of a tubular neighborhood of ℓ.

Proof: It suffices to prove the result in a local model $U \cong T^*S^n$. The obstruction lies in $H^1(U, U \setminus S^n) = 0$ (since $n \geq 2$). The grading is unique once we specify that it should be identity outside a tubular neighborhood of S^n.

Lemma For this preferred grading $\tilde{\tau}_\ell$, we have

$$\tilde{\tau}_\ell(\tilde{\ell}) = \tilde{\ell}[1-n] \quad \text{for any grading } \tilde{\ell} \text{ of } \ell.$$

The proof is an explicit computation in the local model U.
(see "Graded Lagrangian Submanifolds" Lemma 5.7)

Applications:

Thm Let (M^{2n}, ω) satisfy $[\omega] = 0$ and $2c_1(M, \omega) = 0$, n even. Assume M is compact with contact type boundary. Assume M contains an A_3-configuration of Lagrangian spheres L_0, L_1, L_2

Then the Lagrangians $L_1^{(k)} = \tau_{L_2}^{2k}(L_1)$ are pairwise non-Lagrangian-isotopic
(Although they are all smoothly isotopic)

Proof We will only prove the symplectic part of the statement here. When $n=2$, the smooth part follows from the fact that $\tau_{L_2}^{2k} \simeq Id$ smoothly.

Let $\{x_0\} = L_0 \cap L_1 \qquad \{x_1\} = L_1 \cap L_2$

Choose Λ^∞ on M and gradings $\tilde{L}_0, \tilde{L}_1, \tilde{L}_2$ such that

$$\tilde{\mu}(\tilde{L}_0, \tilde{L}_1; x_0) = \tilde{\mu}(\tilde{L}_1, \tilde{L}_2) = 0$$

Set $\tilde{L}_1^{(k)} = \tilde{\tau}_{L_2}^{2k}(\tilde{L}_1) \qquad \tilde{\tau}_{L_2}$ natural grading.

Since $H^1(L_i) = 0$ L_i are also exact, so Floer homologies are well defined and invariant under Lagrangian isotopies.

Now $\quad HF^*(\tilde{L}_0, \tilde{L}_1) = \mathbb{Z}/2\,[0]$
$\quad\quad HF^*(\tilde{L}_1, \tilde{L}_2) = \mathbb{Z}/2\,[0]$

Also $L_0 \cap L_2 = \emptyset$, so we can assume $L_0 \cap L_1^{(k)} = \{x_0\}$

So $HF^*(\tilde{L}_0, \tilde{L}_1^{(k)}) = \mathbb{Z}/2\,[0] \quad \forall k$

Now $HF^*(\tilde{L}_1^{(k)}, \tilde{L}_2) = HF^*(\tilde{L}_1, \tau_{L_2}^{-2k}(\tilde{L}_2))$

$$= HF(\tilde{L}_1, \tilde{L}_2[2k(n-1)]) = \mathbb{Z}/2\,[2k(1-n)]$$

If $\tilde{L}_1^{(k)} \simeq \tilde{L}_1[r]$ then we get $r=0$ and $r = 2k(1-n)$ so $k=0$.