The Action functional and Calculus of variations

First and second variations of the action functional

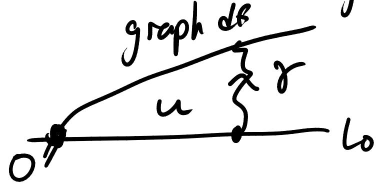
Consider the local situation near a transverse intersection of two Lagrangians L_0, L_1

By Darboux-Weinstein, we may assume $M = T^*\mathbb{R}^n, \sum dp_i \wedge dq^i$
$L_0 = 0\text{-section } \mathbb{R}^n$

$L_1 = \text{graph } df$ for some $f: \mathbb{R}^n \to \mathbb{R}$ $\quad df(0) = 0$
$\text{Hess}_0 f > 0$

Let γ be a path $L_0 \to L_1$
$\gamma : [0,1] \to M$, $\gamma(i) \in L_i$ for $i = 0,1$

Define the action of γ as follows: choose a disk $u: D^2 \to M$ joining γ to the origin, with boundary on $L_0 \cup L_1 \cup \gamma$

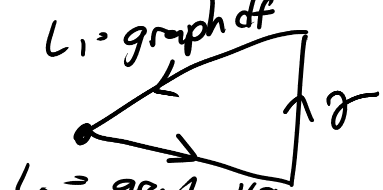


such that boundary orientation agrees with orientation of γ
set

$$\mathcal{A}(\gamma) = \int_{D^2} u^*\omega \quad \text{the symplectic area of this disk.}$$

Now $\omega = \sum dp_i \wedge dq^i = d\left(\sum p_i dq^i\right) = d\lambda$

By stokes $\mathcal{A}(\gamma) = \int_{\partial D^2} u^*\lambda$



Over γ get $\displaystyle\int_{[0,1]} \gamma^* \lambda$

Over $L_0 = \text{graph}\, d(\emptyset)$ λ vanishes $\Rightarrow$ get 0.

Over $L_1 = \text{graph}\, df$ $\displaystyle\int_{\gamma(1)}^{\text{origin}} \lambda = \int_{\pi(\gamma(1))}^{\text{origin}} df = f(0) - f(\pi(\gamma(1)))$

This is computed in $\mathbb{R}^n$, the base of $\pi: T^*\mathbb{R}^n \to \mathbb{R}^n$

Thus $\displaystyle \mathcal{A}(\gamma) = \int_{D^2} u^* \omega = \underbrace{\int_{[0,1]} \gamma^* \lambda - f(\pi(\gamma(1))) + f(0)}_{\text{Doesn't depend on choice of disk, as long as disk lies entirely in Darboux-Weinstein nbhd.}}$

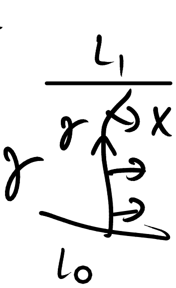
doesn't depend on choice of f or Darboux coords

First variational formula for $\mathcal{A}$

Let's use $\displaystyle \mathcal{A}(\gamma) = \int_{D^2} u^* \omega$ form

A variation of γ is a path γ_s, $s \in (-\varepsilon, \varepsilon)$, $\gamma_0 = \gamma$

Note we get vector field $X = \frac{d}{ds}\big|_{s=0} \gamma_s$ which is VF along γ

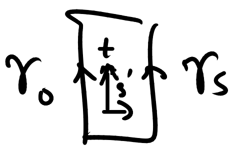


$$X(0) \in T_{\gamma(0)} L_0 \qquad X(1) \in T_{\gamma(1)} L_1$$

We compute $\displaystyle \frac{\delta \mathcal{A}}{\delta \gamma} := \frac{d}{ds}\Big|_{s=0} \mathcal{A}(\gamma_s)$

Let $\quad v(s,t) = \gamma_s(t) \quad$ a surface in M

Difference $\quad A(\gamma_s) - A(\gamma_0) = \displaystyle\int_{\substack{t\in[0,1]\\ s'\in[0,s]}} v^*\omega$



$$= \int_{\substack{t\in[0,1]\\ s'\in[0,s]}} \omega\left(\frac{\partial v}{\partial s'}, \frac{\partial v}{\partial t}\right) ds' \wedge dt = \int_0^1\int_0^s \omega\left(\frac{\partial \gamma_{s'}}{\partial s'}, \dot{\gamma}_{s'}\right) ds'\,dt$$

Apply $\left.\dfrac{d}{ds}\right|_{s=0}$ and FTC $\quad \dfrac{\delta A}{\delta \gamma} = \displaystyle\int_0^1 \omega(X, \dot{\gamma})\,dt$

Critical points are when $\dfrac{\delta A}{\delta\gamma} = 0$ $\forall$ variations. This requires $\dot{\gamma} \equiv 0$.

i.e. γ is a constant path. In our local model, this must be the constant path at the origin.

Second variational formula for A:

Now suppose we are at a critical point, i.e. $\gamma \equiv$ origin.

Differentiating $\quad \dfrac{d}{ds} A = \displaystyle\int_0^1 \omega\left(\frac{\partial\gamma_s}{\partial s}, \dot{\gamma}_s\right) dt \quad$ w.r.t. s

we get $\quad \dfrac{d^2}{ds^2} A = \displaystyle\int_0^1 \omega\left(\frac{\partial^2\gamma_s}{\partial s^2}, \underset{\overset{\|}{0}}{\dot{\gamma}_s}\right) dt + \int_0^1 \omega\left(\frac{\partial\gamma_s}{\partial s}, \frac{\partial\dot{\gamma}_s}{\partial s}\right) dt$

Now $\quad \left.\dfrac{\partial\gamma_s}{\partial s}\right|_{s=0} = X \qquad \left.\dfrac{\partial\dot{\gamma}_s}{\partial s}\right|_{s=0} = \dot{X}$

So $\quad \dfrac{\delta^2 A}{\delta\gamma^2} = \dfrac{d^2}{ds^2}A\Big|_{s=0} = \displaystyle\int_0^1 \omega(X,\dot{X})\,dt$

where $\quad X = \delta\gamma = \dfrac{\partial\gamma_s}{\partial s}$

We can obtain the Hessian of A via polarization

$$\mathrm{Hess}\,A\,(X,Y) = \frac{1}{2}\int_0^1 \big(\omega(X,\dot{Y}) + \omega(Y,\dot{X})\big)\,dt$$

these terms equal

$\Rightarrow$ $\mathrm{Hess}\,A(X,Y) = \displaystyle\int_0^1 \omega(X,\dot{Y})\,dt$, which is symmetric

Note $\omega(X,\dot{Y}) - \omega(Y,\dot{X}) = \omega(X,\dot{Y}) + \omega(\dot{X},Y)$

$= \dfrac{d}{dt}\,\omega(X,Y)$ is a total derivative

So $\displaystyle\int \omega(X,\dot{Y})\,dt - \int \omega(Y,\dot{X})\,dt = \int \frac{d}{dt}\omega(X,Y)\,dt$

$= \omega(\underbrace{X(1),Y(1)}_{\in TL_1}) - \omega(\underbrace{X(0),Y(0)}_{\in TL_0}) = 0$ since L_0, L_1 Lagrangian.

Summary $\quad dA_\gamma(X) = \displaystyle\int_0^1 \omega(X,\dot{\gamma})\,dt$

Crit A = constant paths at intersection points

$\mathrm{Hess}_\gamma A\,(X,Y) = \displaystyle\int_0^1 \omega(X,\dot{Y})\,dt$

Now we get the metric and complex structure involved.

$$g(X,Y) = \omega(X, JY)$$

on vector fields along a path $X(t), Y(t)$

$$\langle X, Y \rangle = \int_0^1 g(X,Y)\,dt$$

$$= \int_0^1 \omega(X, JY)\,dt$$

$$dA_\gamma(X) = \langle X, \nabla A_\gamma \rangle = \int_0^1 \omega(X, J\nabla A_\gamma)\,dt$$

$$= \int_0^1 \omega(X, \dot\gamma)\,dt \qquad \text{so} \quad J\nabla A_\gamma = \dot\gamma$$

$$\nabla A_\gamma = -J\dot\gamma$$

Formally, positive Gradient flow $\frac{\partial}{\partial s}\gamma_s = \nabla A_{\gamma_s} = -J\dot\gamma$

$$\frac{\partial \gamma_s}{\partial s} + J\dot\gamma = 0$$

if $u(s,t) = \gamma_s(t)$ $\quad \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0 \quad$ J-hol curve.

~~Hessian~~

$$\mathrm{Hess}\,\mathcal{A}(X,Y) = \langle X, DY \rangle = \int g(X,DY)\,dt$$

$$\mathrm{Hess}\,\mathcal{A}(X,Y) = \int \omega(X,\dot{Y})\,dt \qquad\qquad \int g(X,DY)\,dt = \int \omega(X, JDY)\,dt$$

So $\quad \dot{Y} = JDY \qquad$ or $\quad DY = -J\dot{Y}$

D is the 1-d Dirac operator $\quad -i\frac{d}{dt} \qquad D^2 = -\frac{d^2}{dt^2} = -\Delta$

This self-adjoint operator represents the Hessian of $\mathcal{A}$ w.r.t. $\langle\cdot,\cdot\rangle$

Now we want to understand spectral theory of

$$D = -J\frac{d}{dt} \quad \text{acting on} \quad \left\{ X(t) : [0,1] \to T_pM \;\middle|\; \begin{array}{l} X(0) \in T_pL_0 \\ X(1) \in T_pL_1 \end{array} \right\}$$

The spectrum depends on the relative position of L_0 and L_1

Since $DY = \lambda Y$ is an ODE, we use the standard method.
look for a solution of the form $\quad Y(t) = e^{At}Y_0$

$$DY = -JAe^{At}Y_0 = \lambda e^{At}Y_0 \Rightarrow -JA = \lambda \quad \text{so} \quad A = \lambda J$$

$$\text{so} \quad Y(t) = e^{\lambda J t}Y_0 = \left(\cos(\lambda t)\,\mathrm{Id} + \sin(\lambda t)\,J\right)Y_0$$

We need $Y(0) = Y_0 \in T_pL_0$ and $Y(1) = e^{\lambda J}Y_0 \in T_pL_1$

Thinking of T_pM as $\mathbb{C}$-vector space via J, $e^{\lambda J}$ is a $\mathbb{C}$-scalar.

So we ask, are there vectors $Y_0 \in T_p L_0$ and scalar $\mu \in U(1)$ such that $Y(1) = \mu Y_0 \in T_p L_1$

Linear algebra:

$$\begin{array}{lll} T_pM & \mathbb{C}^n & \\ T_pL_0 & L_0 = \mathbb{R}^n & \\ T_pL_1 & L_1 = A\mathbb{R}^n & A \in U(n)/O(n) \end{array}$$

seek $v \in \mathbb{R}^n$ such that $\mu v \in L_1 = A\mathbb{R}^n \Leftrightarrow \mu A^{-1} v \in \mathbb{R}^n$

Same as asking that μL_0 and L_1 are not transverse.

The number of solutions is the maslov index of the loop $\mu \mapsto \mu L_0$ in the Lagrangian Grassmannian Λ_n

$= \text{degree of } \det^2 : U(n)/O(n) \to U(1)$

$$\det^2(\mu I) = \mu^{2n}$$

So are $2n$ solutions.

Now each μ as above gives infinitely many eigenvalues λ for $D = -J\frac{d}{dt}$

Since $\mu = e^{\lambda i}$, the spectrum has a symmetry

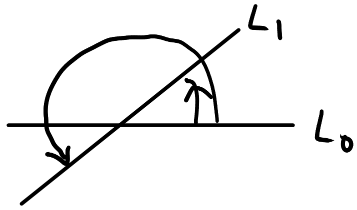
$$\lambda \to \lambda + 2\pi k$$

We conclude that D has $2\pi\mathbb{Z}$-families of eigenvalues, with the number of families being $2n$.

In particular D has ∞-ly many positive + negative eigenvalues!

$\lambda = 0$ is eigenvalue $\Leftrightarrow$ $\mu = 1$ works $\Leftrightarrow$
L_0 and L_1 are not transverse at p.

Eg $n = 1$



$L_1 = e^{i\theta} L_0$
or $L_1 = e^{i(\theta+\pi)} L_0$
or $L_1 = e^{i(\theta+2\pi)} L_0$

$$\mathrm{Spec}\, D = \left\{ \begin{matrix} \ldots, \theta-2\pi, & \theta, & \theta+2\pi, & \theta+4\pi, \ldots \\ \ldots, \theta-\pi, & \theta+\pi, & \theta+3\pi, & \theta+5\pi, \ldots \end{matrix} \right\}$$

We see $2 = 2n$ families with $2\pi\mathbb{Z}$-symmetry.