Almost complex structures on Symplectic manifolds

Recall linear structures on $\mathbb{R}^{2n} = \mathbb{C}^n$

Hermitian inner product $h(u,v) = \sum_{i=1}^{n} u_i \overline{v_i}$

$$h(u,v) = g(u,v) - i\,\omega(u,v)$$

$$\underset{\substack{\parallel \\ -i g(u,v) - \omega(u,v)}}{-ih(u,v)} = h(u,iv) = g(u,iv) - i\,\omega(u,iv)$$

so $\omega(u,v) = -g(u,iv)$ and $g(u,v) = \omega(u,iv)$

g symmetric $\Rightarrow$

$$\omega(iu,iv) = -g(iu,v) = -g(v,iu) = -\omega(v,u) = \omega(u,v)$$

<u>Abstract these properties</u>: (V,ω) symplectic. $J: V \to V$ is <u>a complex structure</u> if $J^2 = -Id$

- J is <u>tame</u> (tamed by ω) if $\omega(x, Jx) > 0 \ \forall x \neq 0$
- J is <u>compatible</u> (with ω) if J is tame and $\omega(x, Jy)$ is symmetric.

When J is compatible with ω, $g(x,y) = \omega(x, Jy)$ is a metric on V. Then (V, ω, J, g) is isomorphic to $\mathbb{C}^n$ with its standard structures.

When J is merely tame, the symmetrization
$$g(x,y) = \frac{1}{2}\left[\omega(x,Jy) + \omega(Jx,y)\right]$$ is a metric on V.

Reason for considering both. Often, we need to perturb J to a "generic" situation. Positivity of $\omega(x, Jx)$ is extremely important, but symmetry is less important. Considering tame ACS's gives us more freedom.

We use the same terminology for structures on a manifold (M, ω) symplectic manifold. $J : TM \to TM$ is called an almost complex structure if $J^2 = -Id$ (at each pt). It is tame/compatible if it is so at each point.

Reason for "almost": A "true" complex manifold has local holomorphic coordinate charts $(z_1, \ldots, z_n) \in U \subseteq \mathbb{C}^n$. The tangent spaces are also identified with $\mathbb{C}^n$, and we the standard structure $J = i$ there. The coordinate transitions preserve this structure, so we get a well-defined J on all of M. The J obtained this way satisfies an extra condition called Integrability.

The Nijenhuis tensor

$$N(X,Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$$

vanishes for all vector fields X, Y.

Almost complex = We don't impose this condition.

Vanishing of N is sufficient for existence of local holomorphic coordinates (Newlander-Nirenberg)

In dimension 2, N vanishes for dimensional reasons.

$\therefore$ Any almost complex surface is a Riemann surface.

Homework: if (M, J) is almost-complex, and $f: M \to \mathbb{C}$ is holomorphic, meaning $df \circ J = i\, df$, then

$$df\,(N(X,Y)) = 0$$

for all vector fields X and Y.

So in general an almost complex manifold may have no holomorphic functions. But it still has many holomorphic curves

$$u : (\Sigma, j) \longrightarrow (M, J) \qquad du \circ j = J \circ du$$

(Σ, j) a Riemann surface

Contractibility of space of tame / compatible almost complex structures.

Linear theory:

$(\mathbb{C}^n, J_0 = i, g, \omega)$ standard complex vector space

$\mathcal{J}_t(\omega)$ = tame ACS $\qquad \mathcal{J}_c(\omega)$ = compatible ACS

Proposition The map $J \mapsto S := (J + J_0)^{-1}(J - J_0)$ is a diffeomorphism of $\mathcal{J}_t(\omega)$ onto the unit ball (w.r.t. g) in space of matrices satisfying the linear equation $J_0 S + S J_0 = 0$

The subspace $\mathcal{J}_c(\omega)$ maps diffeomorphically onto the set of symmetric matrices within this space

$$\mathcal{J}_t(\omega) \overset{\cong}{\longleftrightarrow} \{ S \mid \|S\| < 1 \text{ and } J_0 S + S J_0 = 0 \}$$

$$\mathcal{J}_c(\omega) \longleftrightarrow \{ S \mid \text{" \quad " \quad " and } S^T = S \}$$

(M. Audin attributes this to B. Sévennec)

Proof: let $J \in \mathcal{J}_t(\omega)$ then

$$\omega(x,(J+J_0)x) = \omega(x,Jx) + \omega(x,J_0 x) > 0$$

so $J + J_0$ cannot have a kernel.
Thus $S = (J+J_0)^{-1}(J-J_0)$ is well defined.

Write $A = J_0^{-1}J$
Then $S = [J_0(A+I)]^{-1}[J_0(A-I)] = (A+I)^{-1}(A-I)$

To show $\|S\| < 1$, show $\|Ax-x\|^2 < \|Ax+x\|^2 \quad \forall x$

$$\|Ax+x\|^2 - \|Ax-x\|^2 = g(Ax+x, Ax+x) - g(Ax-x, Ax-x)$$

$$= g(Ax,Ax) + g(Ax,x) + g(x,Ax) + g(x,x)$$

$$-[g(Ax,Ax) - g(Ax,x) - g(x,Ax) + g(x,x)]$$

$$= 4g(x,Ax) = 4g(x, J_0^{-1}Jx) = 4\omega(x, J_0 J_0^{-1} J x)$$

$$= 4\omega(x, Jx) > 0$$

This proves $\|S\| < 1$

Conversely, if $\|S\| < 1$ $\quad I - S$ is invertible
Solve $S = (J+J_0)^{-1}(J-J_0)$ for J

$$(J+J_0)S = J - J_0$$
$$JS + J_0 S = J - J_0$$
$$JS - J = -J_0 S - J_0$$
$$-J(I-S) = -J_0(I+S)$$

$$J = J_0(I+S)(I-S)^{-1}$$

And conversely J satisfies $\omega(x, Jx) > 0$

To see this: $\omega(x, Jx) = g(x, J_0^{-1} Jx) =$

$$= g(x, (I+S)(I-S)^{-1}x) \qquad y = (I-S)^{-1}x$$

$$= g((I-S)y, (I+S)y)$$

$$= g(y - Sy, y + Sy) = g(y,y) - g(Sy,y) + g(y,Sy) - g(Sy,Sy)$$

$$= g(y,y) - g(Sy,Sy) = \|y\|^2 - \|Sy\|^2 > 0 \text{ since } \|S\| < 1$$

Now we check $J^2 = -I \iff J_0 S + S J_0 = 0$

$$J = J_0(I+S)(I-S)^{-1}$$

$$J^2 = -I \iff J_0(I+S)(I-S)^{-1}J_0(I+S)(I-S)^{-1} = -I$$

$$\iff (I+S)\left(\sum_{k=0}^{\infty} S^k\right) J_0(I+S) = J_0(I-S)$$

$$\iff \left(I + 2\sum_{k=1}^{\infty} S^k\right) J_0(I+S) = J_0(I-S)$$

$$\iff J_0 + J_0 S + 2\sum_{k=1}^{\infty} S^k J_0 + 2\sum_{k=1}^{\infty} S^k J_0 S = J_0(I-S)$$

$$\iff 2\sum_1^{\infty} S^k J_0 + 2\sum_1^{\infty} S^k J_0 S = -2J_0 S$$

$$\iff \sum_1^{\infty}\left(S^k J_0 + S^k J_0 S\right) = -J_0 S$$

$$\iff \sum_1^{\infty} S^k J_0 + \sum_0^{\infty} S^k J_0 S = 0$$

$$\iff \sum_1^{\infty} S^k J_0 + (I-S)^{-1} J_0 S = 0$$

$$\Leftrightarrow \left((I-S)^{-1} - I \right) J_0 + (I-S)^{-1} J_0 S = 0$$

$$\Leftrightarrow \left(I - (I-S) \right) J_0 + J_0 S = 0$$

$$\Leftrightarrow \quad S J_0 + J_0 S = 0$$

<u>Simplification</u>:

$$(I+S)(I-S)^{-1} = I + 2\left((I-S)^{-1} - I\right)$$
$$= 2(I-S)^{-1} - I$$

$$J = J_0 (I+S)(I-S)^{-1} = J_0 \left(2(I-S)^{-1} - I \right)$$

$$J^2 = J_0 \left(2(I-S)^{-1} - I \right) J_0 \left(2(I-S)^{-1} - I \right)$$
$$= 4 J_0 (I-S)^{-1} J_0 (I-S)^{-1} - 2 J_0 (I-S)^{-1} J_0$$
$$-2 J_0^2 (I-S)^{-1} + J_0^2 = -I$$

$$J^2 = -I \Leftrightarrow 4 J_0 (I-S)^{-1} J_0 (I-S)^{-1} - 2 J_0 (I-S)^{-1} J_0 - 2 J_0^2 (I-S)^{-1} = 0$$

$$\Leftrightarrow 2 (I-S)^{-1} J_0 (I-S)^{-1} - (I-S)^{-1} J_0 - J_0 (I-S)^{-1} = 0$$
$$\Leftrightarrow 2 (I-S)^{-1} J_0 - (I-S)^{-1} J_0 (I-S) - J_0 = 0$$
$$\Leftrightarrow 2 J_0 - J_0 (I-S) - (I-S) J_0 = 0$$
$$\Leftrightarrow 2 J_0 - J_0 + J_0 S - J_0 + S J_0 = 0$$
$$\Leftrightarrow \quad J_0 S + S J_0 = 0$$

Last we check $\omega(x, Jy)$ symmetric $\Leftrightarrow$ S symmetric

Let $b(x,y) = \omega(x, Jy)$ $\quad g(x,y) = \omega(x, J_0 y)$

$$J = J_0 (I+S)(I-S)^{-1}$$

So $b(x,y) = \omega\left(x, J_0 (I+S)(I-S)^{-1} y\right) = g\left(x, (I+S)(I-S)^{-1} y\right)$

Now $b(x,y)$ is symmetric iff $b((I-S)x,(I-S)y)$ is symmetric. The latter equals:

$$g((I-S)x,(I+S)y) = g(x-Sx, y+Sy)$$

$$= \underset{\text{symmetric}}{g(x,y)} + \underbrace{g(x,Sy) - g(Sx,y)}_{\text{skew-symmetric}} + \underset{\text{symmetric}}{g(Sx,Sy)}$$

This is symmetric iff $g(x,Sy) - g(Sx,y) = 0$

i.e. $g(x,Sy) = g(Sx,y) = g(x,S^Ty)$

This is equivalent to $S = S^T$. ∎

<u>Corollary</u> $\mathcal{J}_t(\omega)$ and $\mathcal{J}_c(\omega)$ are diffeomorphic to convex subsets of vector spaces. Hence they are contractible

<u>Proof</u> For $\mathcal{J}_t(\omega)$ the vector space is $\{S \in Mat_{2n}(\mathbb{R}) \mid J_0 S + S J_0 = 0\}$

For $\mathcal{J}_c(\omega)$ it is $\{S \in Mat_{2n}(\mathbb{R}) \mid J_0 S + S J_0 = 0 \text{ and } S = S^T\}$

The convex set is $\|S\| < 1$ in both cases. ∎

Now we consider a manifold (M,ω). There are fiber bundles of tame or compatible almost complex structures.

$$\begin{array}{ccc} \mathcal{J}_t(\omega_p) & \hookrightarrow & \underline{J}_t \\ \downarrow & & \downarrow \\ p & \in & M \end{array} \qquad \begin{array}{ccc} \mathcal{J}_c(\omega_p) & \hookrightarrow & \underline{J}_c \\ \downarrow & & \downarrow \\ p & \in & M \end{array}$$

Both smooth sub-fiber bundles of

$$\begin{array}{c} \underline{End}(TM) \\ \downarrow \\ M \end{array}$$

To see they really are manifolds, pick Darboux coordinates, standard J_0 in those coordinates, and apply proposition parametrically.

So in each case we have a fibration of manifolds with contractible fibers.

Therefore both $\underline{J}_t$ and $\underline{J}_c$ have sections, and their spaces of sections are in fact contractible.

Denote their spaces of smooth sections by
$$\mathcal{J}_t(M,\omega) = \Gamma(\underline{J}_t) \qquad \mathcal{J}_c(M,\omega) = \Gamma(\underline{J}_c)$$

lesson: because the space of tame/compatible ACS is contractible, from a topological viewpoint we can always assume that (M,ω) also is equipped with some ACS J, and we don't necessarily care that much how it is chosen. (Though we will care for the purposes of analysis).

Local theory of J-holomorphic curves

Let (S,j) be a Riemann surface. Can just think of $D \subset \mathbb{C}$ the closed unit disk for now.

Let (M,J) be a manifold with ACS J.

A function $f: S \to M$ is J-holomorphic if
$$df \circ j = J \circ df$$
That is, if the differential df is complex linear. w.r.t. j on source, J on target.

We can reuse the trick to study J-holomorphic curves locally. Let (M,ω,J) be tame. In a local Darboux coordinate patch U in M, introduce the standard complex structure J_0

J_0 is integrable in U, and it makes U look like an open set in $\mathbb{C}^n$ with coords $(z_1, \dots, z_n)$. We consider maps $f: D \to U$

J_0-holomorphic maps $f: \underset{\mathbb{C}}{\underset{\cap}{D}} \to \underset{\mathbb{C}^n}{\underset{\cap}{U}}$ are entirely classical.

They are vector-valued holomorphic functions

In coordinates $z = x + iy$ on D, introduce operators

$$\partial f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

this i is really J_0.

$$\bar\partial f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Observe that f is J_0-holomorphic iff $\bar\partial f = 0$.

There is a classical parametrix for the $\bar\partial$ operator on D:

$$Pg(z) = \frac{1}{2\pi i}\iint_D \frac{g(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta \qquad \text{for } g \in C^0(D)$$

[P defines a map $L^2(D) \to L^2_1(D)$]

Then $\bar\partial \circ P = \mathrm{Id}$ (and $P \circ \bar\partial = \mathrm{Id}$ when restricted to functions with compact support in $\mathrm{int}(D)$)

In our chart U, $J(p) + J_0(p)$ is invertible because both are tame. So we may define $\sigma(p) = (J + J_0)^{-1}(J - J_0)$ at each point (cf proposition).

The equation $df \circ i = J \circ df$ boils down to

$$\frac{\partial f}{\partial y} = J(f)\frac{\partial f}{\partial x}$$

Now use
$$\frac{\partial f}{\partial y} = \frac{i}{2}(\partial f - \bar{\partial} f) \overset{\text{really}}{=} \frac{J_0}{2}(\partial f - \bar{\partial} f)$$
$$\frac{\partial f}{\partial x} = \frac{1}{2}(\partial f + \bar{\partial} f)$$

$$\begin{aligned}
(J\text{-hol}) &\iff J_0(f)(\partial f - \bar{\partial} f) = J(f)(\partial f + \bar{\partial} f) \\
&\iff J_0 \partial f - J_0 \bar{\partial} f = J\partial f + J\bar{\partial} f \\
&\iff (J_0 - J)\partial f = (J_0 + J)\bar{\partial} f \\
&\iff \underbrace{(J_0 + J)^{-1}(J_0 - J)\partial f}_{= -\sigma(f)\partial f} = \bar{\partial} f \\
&\iff \bar{\partial} f + \sigma(f)\partial f = 0
\end{aligned}$$

(Somewhat similar to Beltrami equation $\bar{\partial} f = \mu(z)\partial f$).

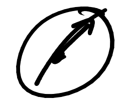
<u>Key observation</u>: $\bar{\partial} f + \sigma(f)\partial f = 0$ is equivalent to $\bar{\partial} g = 0$ where $g = f + P\sigma(f)\partial f$ (since $\bar{\partial} \circ P = Id$)

This means that in a local chart, J-holomorphic maps $f : D \to U$ are equivalent to ordinary (J_0-) holomorphic maps known from complex analysis.

The local theory of J-holomorphic curves tracks closely the theory for ordinary complex analytic curves.

(M, J) almost complex

<u>Theorem</u> $\forall p \in M$, and $\forall v \in T_pM$ sufficiently small
$\exists$ J-holomorphic $f : (D, 0) \to (M, p)$
such that $df_0(1) = v$



<u>Theorem</u> The critical points of a J-holomorphic map are isolated.

etc.