Partial Differential Equations Review

Standard problems

1) Heat eqn on rod of length L with fixed temp at ends

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad u(0,t)=0, \quad u(L,t)=0$$

Initial condition $u(x,0)=f(x)$

Tip: You will end up using a sine series in this problem

2) Heat eqn with insulated ends

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial x}(0,t)=0, \quad \frac{\partial u}{\partial x}(L,t)=0$$

Initial condition $u(x,0)=f(x)$

Tip: You will end up using a cosine series in this problem.

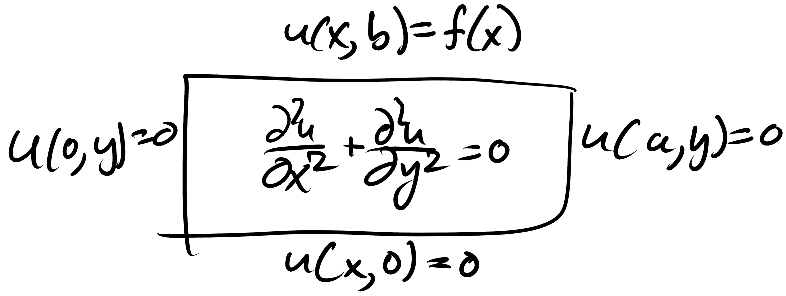
3) Wave eqn on string of length L with fixed ends.

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad u(0,t)=0, \quad u(L,t)=0$$

Initial conditions $u(x,0)=f(x)$ $\quad \frac{\partial u}{\partial t}(x,0)=g(x)$

Tip: you will use a sine series to satisfy both initial conditions, but in the series for $g(x)$ you will need to divide by something (see lecture 35)

4) Laplace's equation on a rectangle w/ 3 homogeneous boundaries



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(0,y) = 0$
$u(a,y) = 0$
$u(x,0) = 0$
$u(x,b) = f(x)$

Tip: you will use a sine series to satisfy the Nonhomogeneous boundary conditions.

** For these "standard problem", I expect you to be familiar with all aspects of their solution. **

There may also be questions about "less than standard" or "slightly exotic" problems.

1) Different boundary conditions

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = 0 \qquad \frac{\partial u}{\partial x}(10,t) = 0$$

<u>Problem</u>: Find general solution.

2) Nonhomogeneous boundary conditions

$$\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = 0 \qquad u(1,t) = 1$$

Initial conditions: $u(x,0) = x$, $\frac{\partial u}{\partial t}(x,0) = \sin \pi x$

<u>Problem</u> (a) Find steady state solution u_0 to Wave eqn + B.C.
(b) Find the conditions that $w = u - u_0$ must satisfy
(c) solve for w, hence for u

3) More exotic variations:

Eigenvalue problem for $y(x)$

$$y'' + \lambda y = 0 \quad , \quad y'(0) = 0 \quad , \quad y(1) + y'(1) = 0$$

Show that the <u>positive</u> eigenvalues are precisely the solutions of the equation

$$\sqrt{\lambda} \tan \sqrt{\lambda} = 1$$

<u>Solutions</u>: 1) $\frac{\partial u}{\partial t} = 5 \frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t) = 0 \quad \frac{\partial u}{\partial x}(10,t) = 0$

Separable solutions: $u(x,t) = X(x) T(t)$

$$\frac{\partial u}{\partial t} = 5 \frac{\partial^2 u}{\partial x^2} \longrightarrow \frac{dT}{dt} X = 5 T \frac{d^2 X}{dx^2} \Rightarrow \frac{1}{5T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2}$$

$$\longrightarrow \frac{1}{5T} \frac{dT}{dt} = -\lambda = \frac{1}{X} \frac{d^2 X}{dx^2} \Rightarrow \frac{d^2 X}{dx^2} + \lambda X = 0$$

$$\frac{dT}{dt} = -5\lambda T$$

$$u(0,t) = 0 \Longrightarrow X(0) = 0$$

$$\frac{\partial u}{\partial x}(10,t) = 0 \Longrightarrow \frac{dX}{dx}(10) = 0$$

Eigenvalue problem $\frac{d^2 X}{dx^2} + \lambda X = 0 \quad X(0) = 0 \quad \frac{dX}{dx}(10) = 0$

Eigenvalues: Three cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$

$\lambda = 0$: $\frac{d^2 X}{dx^2} = 0 \qquad X = Ax + B \qquad X(0) = 0 \Rightarrow B = 0$

$\qquad \frac{dX}{dx} = A \qquad \frac{dX}{dx}(10) = 0 \Rightarrow A = 0$

Hence $X = 0$ $\therefore$ $\lambda = 0$ is not an eigenvalue

$\lambda < 0$

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}, \quad X(0) = 0 \Rightarrow A + B = 0$$

$$\frac{dX}{dx} = A\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - \sqrt{-\lambda}Be^{-\sqrt{-\lambda}x}, \quad \frac{dX}{dx}(10) = 0 \Rightarrow Ae^{\sqrt{-\lambda}10} - Be^{-\sqrt{-\lambda}10} = 0$$

get $\quad B = -A \Rightarrow 0 = A\big(\underbrace{e^{\sqrt{-\lambda}10} + e^{-\sqrt{-\lambda}10}}_{\text{this is positive}}\big) \Rightarrow A = 0$

$\Rightarrow B = 0$

so $X = 0$ and $\lambda < 0$ is not an eigenvalue

$\lambda > 0$: $\quad X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$X(0) = 0 \Rightarrow A = 0 \Rightarrow X = B\sin\sqrt{\lambda}x$

$$\frac{dX}{dx} = B\cos\sqrt{\lambda}x \qquad \frac{dX}{dx}(10) = 0 \Rightarrow \cos\sqrt{\lambda}\cdot 10 = 0$$

$$\Rightarrow \sqrt{\lambda}\cdot 10 = \left(n - \tfrac{1}{2}\right)\pi \quad n = 1, 2, 3, \ldots$$

Eigenvalues $\lambda_n = \left[\dfrac{\left(n - \frac{1}{2}\right)\pi}{10}\right]^2 \quad n = 1, 2, 3, \ldots$

Eigenfunctions $X_n = \sin\dfrac{\left(n - \frac{1}{2}\right)\pi x}{10}$

Corresponding $T_n(t) = e^{-5\left[\frac{\left(n - \frac{1}{2}\right)\pi}{10}\right]^2 t}$

Seperable solutions: $u_n(x,t) = e^{-5\left[\frac{\left(n - \frac{1}{2}\right)\pi}{10}\right]^2 t} \sin\dfrac{\left(n - \frac{1}{2}\right)\pi x}{10}$

General solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-5\left[\frac{\left(n - \frac{1}{2}\right)\pi}{10}\right]^2 t} \sin\frac{\left(n - \frac{1}{2}\right)\pi x}{10}$$

2) Nonhomogeneous boundary conditions

$$\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2} \qquad u(0,t)=0 \qquad u(1,t)=1$$

Initial conditions: $u(x,0) = x$, $\frac{\partial u}{\partial t}(x,0) = \sin \pi x$

Problem (a) Find steady state solution u_0 to Wave eqn + B.C.

(b) Find the conditions that $w = u - u_0$ must satisfy

(c) solve for w, hence for u

(a) Steady state u_0: $\frac{\partial u_0}{\partial t} = 0 \Rightarrow \frac{\partial^2 u_0}{\partial x^2} = 0 \Rightarrow u_0 = Ax + B$

$u_0(0,t) = 0 \implies B = 0 \qquad u_0(1,t) = 0 \Rightarrow A = 1$

So $u_0(x,t) = x$ is the steady state solution.

(b) $w = u - u_0 = u - x$ satisfies

$$\frac{\partial^2 w}{\partial t^2} = 25\frac{\partial^2 w}{\partial x^2} \qquad \begin{aligned} w(0,t) &= u(0,t) - 0 = 0 - 0 = 0 \\ w(1,t) &= u(1,t) - 1 = 1 - 1 = 0 \end{aligned}$$

$$w(x,0) = u(x,0) - x = x - x = 0$$

$$\frac{\partial w}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,0) - \frac{\partial u_0}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,0) = \sin \pi x$$

(c) Need to solve $\frac{\partial^2 w}{\partial t^2} = 25\frac{\partial^2 w}{\partial x^2}$

$$w(0,t) = 0$$
$$w(1,t) = 0$$
$$w(x,0) = 0$$
$$\frac{\partial w}{\partial t}(x,0) = \sin \pi x$$

This is the standard wave eqn problem with $c = 5$, $L = 1$

$$\Rightarrow w(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos 5n\pi t + B_n \sin 5n\pi t \right) \sin n\pi x$$

$$0 = w(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x \quad \text{so} \quad \underline{A_n = 0}$$

$$\sin \pi x = \frac{\partial w}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n (5n\pi) \sin n\pi x$$

so $B_n = 0$ for $n = 2, 3, \ldots$

and $B_1 \, 5 \cdot 1 \cdot \pi = 1$ so $B_n = \frac{1}{5\pi}$

Solution: $w(x,t) = \frac{1}{5\pi} \sin 5\pi t \sin \pi x$

$u = w + u_0$: $\boxed{u(x,t) = x + \frac{1}{5\pi} \sin 5\pi t \sin \pi x}$

3) More exotic variations:

Eigenvalue problem for $y(x)$

$$y'' + \lambda y = 0 \; , \quad y'(0) = 0 \; , \quad y(1) + y'(1) = 0$$

Show that the positive eigenvalues are precisely the solutions of the equation

$$\sqrt{\lambda} \tan \sqrt{\lambda} = 1$$

Since the problem only asks about positive eigenvalues, we assume $\lambda > 0$

$\lambda > 0 \qquad y'' + \lambda y = 0 \qquad r^2 + \lambda = 0 \qquad r^2 = \pm i\sqrt{\lambda}$

$$y(x) = A \cos \sqrt{\lambda}\, x + B \sin \sqrt{\lambda}\, x$$

$$y'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$y'(0)=0 \Longrightarrow \quad 0 = y'(0) = B\sqrt{\lambda} \Rightarrow B=0$$

$$\therefore \quad y = A\cos\sqrt{\lambda}x$$
$$y' = -A\sqrt{\lambda}\sin\sqrt{\lambda}x$$

$$y(1)+y'(1)=0 \Longrightarrow A\cos\sqrt{\lambda} - A\sqrt{\lambda}\sin\sqrt{\lambda} = 0$$
$$A(\cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda}) = 0$$

In order for λ to be an eigenvalue, we need $y \neq 0$, hence $A \neq 0$

Therefore we need

$$\cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda} = 0$$
$$\cos\sqrt{\lambda} = \sqrt{\lambda}\sin\sqrt{\lambda}$$

$$1 = \sqrt{\lambda}\,\frac{\sin\sqrt{\lambda}}{\cos\sqrt{\lambda}} = \sqrt{\lambda}\tan\sqrt{\lambda}$$

Thus λ is an eigenvalue $\Longleftrightarrow \sqrt{\lambda}\tan\sqrt{\lambda} = 1$