Population models (examples of Autonomous DEs)

Let t denote time, and let $P(t)$ denote the population of some type of organism. (bacteria, humans, deer, ...)

We introduce

β = birth rate = births per unit time per unit of population.

δ = death rate

Over an interval of time Δt, there will be

$$\beta \cdot P \cdot \Delta t \quad \text{births}$$

and

$$\delta \cdot P \cdot \Delta t \quad \text{deaths}$$

so the change in population will be

$$\Delta P \approx \beta \cdot P \cdot \Delta t - \delta \cdot P \cdot \Delta t = (\beta - \delta) P \Delta t$$

$$\frac{\Delta P}{\Delta t} \approx (\beta - \delta) P$$

the limit as $\Delta t \to 0$, we obtain $\dfrac{dP}{dt} = (\beta - \delta) P$

This is the basic population model. We can vary the parameters β and δ, even allowing them to depend on t or P.

The point is to understand how the behavior of the population over time depends on the choice of parameters β, δ in the construction of the model.

Simplest case: β and δ are constant.

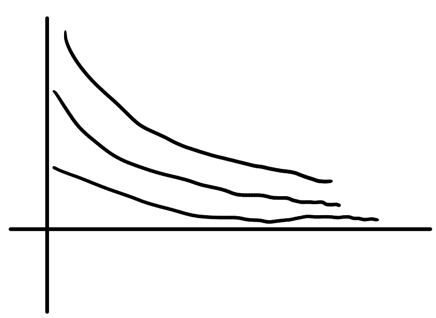
$$\frac{dP}{dt} = kP, \quad \text{where} \quad k = \beta - \delta$$

Solution: $P(t) = P_0 e^{kt}$ where $P_0 = P(0)$ is initial pop.

if $\beta > \delta$ then $k > 0$, and we have exponential growth.

EXPLOSION!

if $\beta < \delta$ then $k < 0$, and we have exponential decay

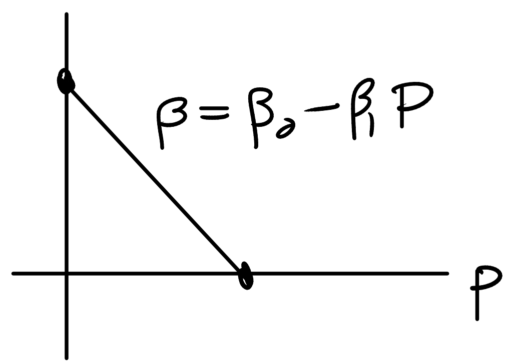


EXTINCTION!

what if β and δ depend on P?

A reasonable model is that the birth rate decreases with increased population. This will be true if there are environmental factors, such as a limited food supply, which limit reproduction at high population levels.

Then $\beta = \beta(P)$ will look like



Negative slope $-\beta_1$
Hits 0 at
$P = \dfrac{\beta_0}{\beta_1}$

Assume Death rate is constant $\delta = \delta_0$

So our population model becomes.

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0)P = (\beta_0 - \delta_0)P - \beta_1 P^2$$
$$= \beta_1 P\left(\frac{\beta_0 - \delta_0}{\beta_1} - P\right)$$

Or $\underbrace{\dfrac{dP}{dt} = kP(M-P)}$ $\qquad k = \beta_1$, $M = \dfrac{\beta_0 - \delta_0}{\beta_1}$

This is called the <u>logistic equation</u>.

Many different situations are modeled by this equation

Let's try to understand this model.
For simplicity take $k = 17$, $M = 256$.

$$\frac{dP}{dt} = 17P(256-P)$$

Separate variables

$$\frac{dP}{P(256-P)} = 17\,dt$$

Integrate $\int 17\,dt = 17t + C$

$$\int \frac{dP}{P(256-P)} = \int \frac{1}{256}\left(\frac{1}{P} + \frac{1}{256-P}\right) dP =$$

$$= \frac{1}{256}\left(\ln|P| - \ln|256-P|\right)$$

So $\frac{1}{256}\left(\ln|P| - \ln|256-P|\right) = 17t + C$

$$\ln|P| - \ln|256-P| = 17\cdot 256\,t + 256C$$

$$\ln\left|\frac{P}{256-P}\right| = 17\cdot 256t + D$$

$$\left|\frac{P}{256-P}\right| = e^{D} e^{(17\cdot 256)t}$$

$$\frac{P}{256-P} = \pm e^{D} e^{(17\cdot 256)t}$$

$$\frac{P}{256-P} = \underset{\substack{\uparrow \\ \text{constant}}}{B}\, e^{(17\cdot256)t}$$

Solve for P:

$$P = 256\,Be^{(17\cdot256)t} - PBe^{(17\cdot256)t}$$

$$\left(1+Be^{(17\cdot256)t}\right)P = 256\,B\,e^{(17\cdot256)t}$$

$$P = \frac{256\,Be^{(17\cdot256)t}}{1+Be^{(17\cdot256)t}}$$

$$= \frac{256}{B^{-1}e^{-(17\cdot256)t}+1}$$

Now If P_0 is population at time 0, we find

$$\frac{P_0}{256-P_0} = B$$

So $$P = \frac{256}{\left(\frac{256-P_0}{P_0}\right)e^{-(17\cdot256)t}+1}$$

$$P = \frac{256\,P_0}{(256-P_0)e^{-(17\cdot256)t}+P_0}$$

For $\dfrac{dP}{dt} = kP(M-P)$

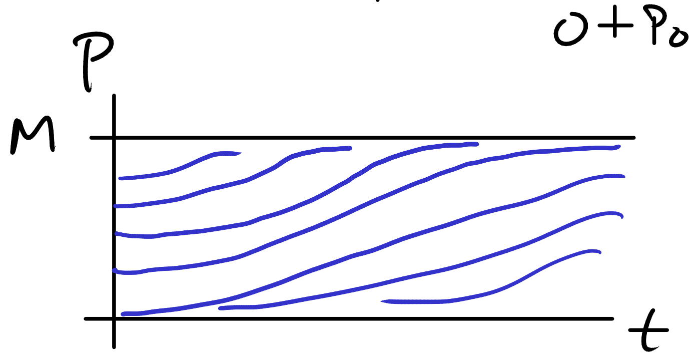
$$P = \frac{MP_0}{(M-P_0)e^{-kMt} + P_0}$$ is the general solution.

Solution curves:

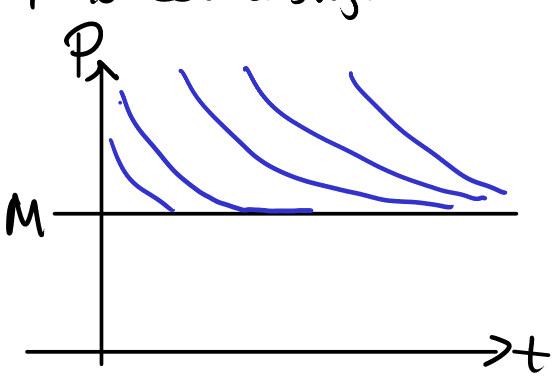
Two cases: if $0 < P_0 < M$ then $M - P_0 > 0$ so $(M-P_0)e^{-kMt}$ is <u>decreasing</u>

and P is <u>increasing</u>.

$t \to +\infty \quad P \to \dfrac{MP_0}{0 + P_0} = M$



If $P_0 > M$ then $M - P_0 < 0$ and $(M-P_0)$ is <u>increasing</u> so P is decreasing.



This reveals the meaning of the constant M:
it is the carrying capacity of the system.

It is the maximum population that is stable.

Q: if the birth rate is $\beta = 100 - P$
and the death rate is $\delta = P + 2$

What is the carrying capacity?

A: $$\frac{dP}{dt} = (\beta - \delta)P = (100 - P - 2 - P)P = (98 - 2P)P$$

$$\frac{dP}{dt} = 2P\left(\frac{98}{2} - P\right)$$

So the carrying capacity is $\frac{98}{2} = 49$