Substitution and other methods.

Just as u-substitution sometimes makes integrals easier, we can use it to solve differential equations

$$\frac{dy}{dx} = (4x+y)^2$$

Try $u = 4x+y$ $\qquad \frac{du}{dx} = 4 + \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{du}{dx} - 4$$

$$\frac{du}{dx} - 4 = u^2 \qquad \frac{du}{dx} = u^2+4$$

$$\int \frac{du}{u^2+4} = \int 1\,dx = x+C \qquad \int \frac{du}{u^2+4} = \frac{1}{2}\arctan\left(\frac{u}{2}\right)$$

So $\frac{1}{2}\arctan\left(\frac{u}{2}\right) = x+C$

$$u = 2\tan(2x+2C)$$

$$4x+y = 2\tan(2x+2C)$$

$$\boxed{y = 2\tan(2x+2C) - 4x.}$$

Sometimes one must manipulate the equation to find a good substitution

$$x^2 \frac{dy}{dx} = xy + x^2 e^{y/x}$$

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

Try $u = \frac{y}{x}$ $\qquad \frac{du}{dx} = \frac{-y}{x^2} + \frac{1}{x}\frac{dy}{dx}$

$$\frac{dy}{dx} = x\frac{du}{dx} + \frac{y}{x} = x\frac{du}{dx} + u$$

$$x\frac{du}{dx} + u = u + e^u$$

$$x\frac{du}{dx} = e^u \longrightarrow \frac{du}{e^u} = \frac{dx}{x}$$

$$-e^{-u} = \ln|x| + C$$

$$-u = \ln(-\ln|x| - C)$$

$$u = -\ln(-\ln|x| - C)$$

$$\frac{y}{x} = -\ln(-\ln|x| - C)$$

$$\boxed{y = -x\ln(-\ln|x| - C)}$$

Bernoulli equation is another standard form with a standard substitution.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

If $n=0$ or $n=1$ this is a linear equation use integrating factor.

Use the substitution $u = y^{1-n}$ for a Bernoulli equation.

Eg $x\frac{dy}{dx} + y = 3xy^{4/3}$

$n = \frac{4}{3}$, $1-n = -\frac{1}{3}$ so use $u = y^{-1/3}$ $y = u^{-3}$

$$\frac{dy}{dx} = -3u^{-4}\frac{du}{dx}$$

$$-3xu^{-4}\frac{du}{dx} + u^{-3} = 3xu^{-4}$$

Divide by $-3xu^{-4}$ $\frac{du}{dx} - \frac{1}{3x}u = -1$

This is a linear equation $\int P(x)dx$ $\int -\frac{1}{3x}dx = -\frac{1}{3}\ln x$

Integrating factor: $\rho(x) = e^{-\frac{1}{3}\ln x} = x^{-\frac{1}{3}}$

$$x^{-\frac{1}{3}}\frac{du}{dx} - \frac{1}{3}x^{-\frac{4}{3}}u = -x^{-\frac{1}{3}}$$

$$\frac{d}{dx}\left(x^{-\frac{1}{3}}u\right) = -x^{-\frac{1}{3}}$$

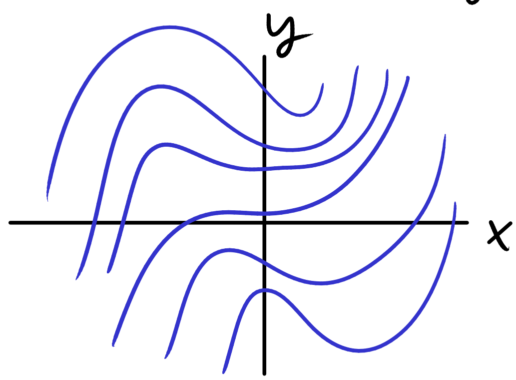
$$x^{-\frac{1}{3}}u = -\frac{3}{2}x^{\frac{2}{3}} + C$$

$$y^{-\frac{1}{3}} = u = -\frac{3}{2}x + Cx^{\frac{1}{3}}$$

$$y = u^{-3} = \left(-\frac{3}{2}x + Cx^{\frac{1}{3}}\right)^{-3}$$

Exact equations:

Consider solution curves of $\frac{dy}{dx} = f(x,y)$



This reminds us of the level curves of a function of two variables $F(x,y)$

Level curves $F(x,y) = C$ $\xleftrightarrow{???}$ Solution curves of $\frac{dy}{dx} = f(x,y)$

We want to find such a function $F(x,y)$ if possible.

In the other direction, what is a differential equation satisfied by y when $F(x,y) = C$?

Take $F(x,y) = C$ and differentiate

$$\frac{d}{dx}\left(F(x,y)\right) = \frac{d}{dx}(C) = 0$$

$$\frac{d}{dx}\left(F(x,y)\right) = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

by multiveriable chain rule, regarding x and y as functions of x.

So $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$ if $F(x,y) = C$

Eg. What is a differential equation for $\frac{dy}{dx}$ satisfied by the level curves of $F(x,y) = x^2y + y^3$?

Differentiate $x^2y + y^3 = C$

$$\frac{d}{dx}\left(x^2y + y^3\right) = (2xy) + \left(x^2 + 3y^2\right)\frac{dy}{dx}$$

So

$$(2xy) + (x^2 + 3y^2)\frac{dy}{dx} = 0 \quad \text{for a level curve.}$$

or, $\frac{dy}{dx} = \frac{-2xy}{x^2+3y^2}$

The real question is to go the other way, from the differential equation to the function $F(x,y)$. See next lecture.