Wave Equation II

Let's recall the problem: Vibrating string

Domain: $0 \le x \le L$ $\qquad$ L = length of string

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{wave eqn.} \\ \left.\begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}\right\} & \text{boundary conditions. Ends of string fixed} \\ u(x,0) = f(x) & \text{initial position} \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) & \text{initial velocity.} \end{cases}$$

This problem may be solved by separation of variables. It is very much analogous to what we did to solve the heat equation. The steps are:

1) Write $u(x,t) = T(t)X(x)$. Figure out the Ordinary DEs that T and X must satisfy in order for u to satisfy the partial DE.

2) Translate Boundary conditions into endpoint conditions on X. You should get an eigenvalue problem for X.

3) Solve the eigenvalue problem: get eigenvalues λ_n eigenfunctions X_n.

4) Find corresponding T_n and $u_n = T_n X_n$.
Now you have found the <u>separable solutions.</u>

5) If the Partial DE and boundary conditions are linear homogeneous, write the general solution
$$u = \sum c_n u_n$$

6) Figure out what the constants have to be in order to satisfy the initial condition(s).

Let's run through this for the wave eqn.

1) Suppose $u(x,t) = T(t)X(t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{d^2 T}{dt^2} \cdot X = T \frac{d^2 X}{dx^2}$$

$$\Rightarrow \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2}$$

One side independent of t, other independent of x, So both are constant.

$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = -\lambda = \frac{1}{X} \frac{d^2 X}{dx^2}$$

$$\frac{d^2 T}{dt^2} + c^2 \lambda T = 0 \quad , \quad \frac{d^2 X}{dx^2} + \lambda X = 0$$

2) Boundary conditions $u(0,t)=0 \Rightarrow X(0)T(t)=0 \Rightarrow X(0)=0$
$u(L,t)=0 \Rightarrow X(L)T(t)=0 \Rightarrow X(L)=0$

So $X(x)$ satisfies
$$\left.\begin{aligned} &\frac{d^2X}{dx^2}+\lambda X=0 \\ &X(0)=0 \\ &X(L)=0 \end{aligned}\right\}$$
A "well-known" eigenvalue problem.

3) Solution: eigenvalues $\left(\frac{\pi}{L}\right)^2, \left(\frac{2\pi}{L}\right)^2, \ldots \quad \lambda_n=\left(\frac{n\pi}{L}\right)^2 \quad n=1,2,3\ldots$

eigenfunctions (Modes of oscillation. $\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots \quad X_n=\sin\frac{n\pi x}{L} \quad n=1,2,3.$

4) T_n must satisfy $\frac{d^2T_n}{dt^2}+c^2\lambda_n T_n=0$
$$\frac{d^2T_n}{dt^2}+c^2\left(\frac{n\pi}{L}\right)^2 T_n=0$$

Solutions: $\sin\frac{cn\pi t}{L}, \quad \cos\frac{cn\pi t}{L}$
$$T_n(t)=A_n\cos\frac{cn\pi t}{L}+B_n\sin\frac{cn\pi t}{L}$$
So $u_n(x,t)=\left(A_n\cos\frac{cn\pi t}{L}+B_n\sin\frac{cn\pi t}{L}\right)\sin\frac{n\pi x}{L}$

5) The conditions $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}, \; u(0,t)=0, u(L,t)=0$

are linear homogeneous, so the <u>general solution</u> is
$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{cn\pi t}{L}+B_n\sin\frac{cn\pi t}{L}\right)\sin\frac{n\pi x}{L}$$

6) Initial conditions

Initial position: $$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \overset{?}{=} f(x)$$

This is the Fourier sine series for $f(x)$.

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Velocity: $$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left[A_n \frac{cn\pi}{L}\left(-\sin\frac{cn\pi t}{L}\right) + B_n \frac{cn\pi}{L} \cos \frac{cn\pi t}{L}\right] \sin \frac{n\pi x}{L}$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \left(B_n \frac{cn\pi}{L}\right) \sin \frac{n\pi x}{L} = g(x)$$

This is a Fourier sine series for $g(x)$, but with a twist.

$B_n \frac{cn\pi}{L}$ = Fourier sine coefficient of $g(x)$ $= \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$

$$B_n = \left(\frac{L}{cn\pi}\right)\frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

Special cases • Plucked string (like guitar) $u(x,0) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = 0$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{cn\pi t}{L} \sin \frac{n\pi x}{L}, \text{ where } A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

• Struck string (like piano) $u(x,0) = 0$

$\frac{\partial u}{\partial t}(x,0) = g(x)$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{cn\pi t}{L} \sin \frac{n\pi x}{L} \text{ where } B_n = \frac{2}{cn\pi}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$