Wave equation I

The wave equation is $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

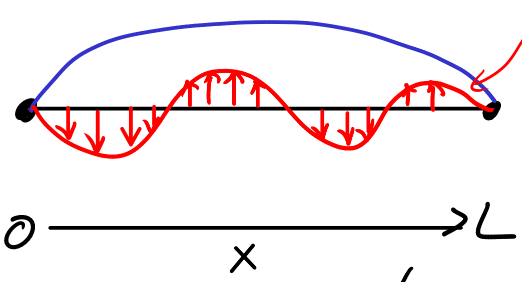
c is the speed of the wave; it depends on the medium.

Free wave propagation

$$u(x,t) = \underbrace{F(x-ct)}_{\text{right moving wave}} + \underbrace{G(x+ct)}_{\text{left moving wave}}$$

These waves have no boundary conditions, and the pass right through each other without interacting. They are "free".

We will be interested in a situation where the waves are <u>confined</u>, such as in a <u>vibrating string</u>:



$u(x,t)$ represents displacement of string from its rest position.

$(L = \text{length})$

The ends of the string are <u>fixed</u>. They don't get displaced. This translates into the boundary conditions

$$u(0,t) = 0 \quad , \quad u(L,t) = 0$$

The function $u(x,t)$ will satisfy the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Since $u(x,t)$ represents the position, $\frac{\partial u}{\partial t}$ is the velocity and $\frac{\partial^2 u}{\partial t^2}$ is the acceleration. So the wave equation says that the acceleration depends on the shape of the string. (Recall that $\frac{\partial^2 u}{\partial x^2}$ measures the convexity/concavity of the graph.)

Since the differential equation is telling us acceleration, we need to specify the initial position and initial velocity in order to get a unique solution.

Initial conditions: $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$

where $f(x)$ describes initial position and $g(x)$ describes initial velocity.

All together, we get a complete wave equation problem

$$\left\{ \begin{array}{ll} \text{Domain } 0 \le x \le L & \\ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, & u(x,0) = f(x), \\ u(0,t) = 0, & \frac{\partial u}{\partial t}(x,0) = g(x). \\ u(L,t) = 0, & \end{array} \right\}$$

For any reasonable functions $f(x)$ and $g(x)$, this problem has a unique solution $u(x,t)$.