Endpoint value problems and eigenvalues

Previously we have studied Initial value problems eg.

$$\begin{cases} ay'' + by' + cy = f(x) & \leftarrow \text{Differential equation} \\ y(0) = b_0 & \\ y'(0) = b_1 & \end{cases} \quad \Big\} \text{ initial conditions}$$

Under reasonable assumptions, this problem always has a unique solution.

Contrast to this the endpoint value problem

$$\begin{cases} ay'' + by' + cy = f(x) & \leftarrow \text{Differential equation} \\ y(0) = b_0 & \\ y(L) = b_1 & \end{cases} \quad \Big\} \text{ endpoint conditions.}$$

We are trying to specify the value of the solution at Different values of x.

Eg. What are the solutions of

$$\begin{cases} y'' + y = 0 \\ y(0) = 0 \\ y(\frac{\pi}{2}) = 7 \end{cases}$$

$y'' + y = 0 \implies y = c_1 \cos x + c_2 \sin x$ for some constant c_1 and c_2

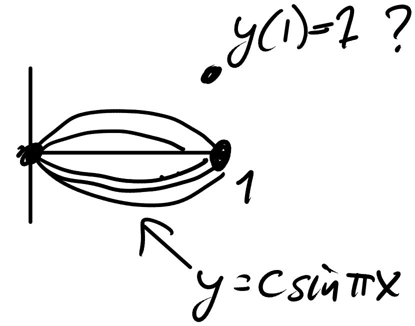
$y(0) = 0 \Rightarrow c_1 \cos 0 + c_2 \sin 0 = 0 \implies c_1 = 0$

$c_1 1 + c_2 \cdot 0 = 0$

$\therefore \quad y = C_2 \sin x$

$y(\frac{\pi}{2}) = 7 \Rightarrow C_2 \sin\frac{\pi}{2} = 7 \Rightarrow C_2 = 7$

So $y = 7\sin x$ is the only solution.



Ex 2: $\begin{cases} y'' + \pi^2 y = 0 \\ y(0) = 0 \\ y(1) = 1 \end{cases}$

$y'' + \pi^2 y \Rightarrow y = C_1 \cos \pi x + C_2 \sin \pi x$

$y(0) = 0 \Rightarrow C_1 \cos 0 + C_2 \sin 0 = 0$

$\Rightarrow C_1 = 0$

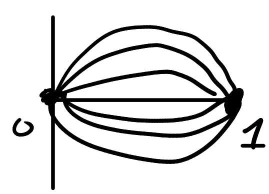
So $y = C_2 \sin \pi x$. Then $y(1) = 1 \Rightarrow C_2 \sin \pi = 1$

$\Rightarrow C_2 \cdot 0 = 1$

$\Rightarrow 0 = 1$

This is absurd, so this endpoint problem has <u>no solution</u>.

But how about $\begin{cases} y'' + \pi^2 y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$



$y = C_1 \cos \pi x + C_2 \sin \pi x$

$C_1 \cos 0 + C_2 \sin 0 = 0$

$C_1 = 0$

So $y = C_2 \sin \pi x$ $\quad y(1) = 0 \Rightarrow C_2 \sin \pi = 0$

This is always true since $\sin \pi = 0$

So this problem has <u>infinitely many solutions</u>:

$\boxed{y = C \sin \pi x \text{ for any constant } c}$

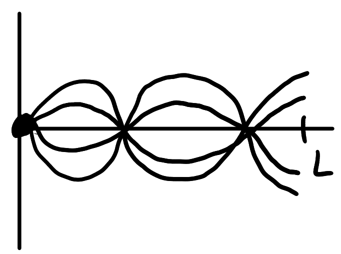
Thus, there is no existence and uniqueness theorem for Endpoint problems.
Instead, the question of whether solutions exist and how many becomes the focus of our study.

Consider the general class of problems

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(L) = 0 \end{cases}$$

Here λ and L are parameters we can vary to get different problems.

One way to think about what this problem is asking: Start with initial condition $y(0)=0$. Choose any initial velocity $y'(0)$ that you want, let $y(x)$ evolve according to the differential equation $y''+\lambda y=0$, and see where it ends up at $x=L$



same initial position, same diff eq, different initial velocities, different $y(L)$ values.

So let's solve as far as possible for as many λ's as possible

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(L) = 0 \end{cases}$$

Case $\lambda = 0$

$$y''=0 \Rightarrow y = c_1 + c_2 x$$
$$y(0)=0 \Rightarrow c_1 = 0 \Rightarrow y = c_2 x$$
$$y(L)=0 \Rightarrow c_2 L = 0 \Rightarrow c_2 = 0 \Rightarrow y = 0$$

So $y \doteq 0$ is the unique solution.

Case $\lambda > 0$: $\quad y'' + \lambda y = 0 \Rightarrow y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$

$y(0) = 0 \Rightarrow c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0$

so $y = c_2 \sin\sqrt{\lambda}x$

$y(L) = 0 \Rightarrow c_2 \sin\sqrt{\lambda}L = 0$

Now either $c_2 = 0$ or $\sin\sqrt{\lambda}L = 0$

$\sin\sqrt{\lambda}L = 0$ happens if $\sqrt{\lambda}L = n\pi$ for some integer n.

$$\sqrt{\lambda} = \frac{n\pi}{L} \qquad \frac{\sqrt{\lambda}L}{\pi} = n$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

So there are two possibilities:

- if $\frac{\sqrt{\lambda}L}{\pi}$ is an integer then there are infinitely many sol'ns

$$y = c_2 \sin\sqrt{\lambda}x = c_2 \sin\frac{n\pi}{L}x$$

- if $\frac{\sqrt{\lambda}L}{\pi}$ is not an integer, then there is only one solution

$$y = 0$$

If $\lambda < 0$, we can show that $\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(L) = 0 \end{cases}$ has only $y = 0$ as a solution.

The numbers $\lambda = \left(\frac{n\pi}{L}\right)^2$ for which there are infinitely many solutions are called the <u>eigenvalues</u>, and the nonzero solutions: $y = \sin\frac{n\pi}{L}x$ are the <u>eigenfunctions</u>.