Forced Oscillation and Fourier Series.

Recall Forced oscillator: $mx'' + cx' + kx = F(t)$

Today, we only consider undamped case $c = 0$

$$mx'' + kx = F(t).$$

We will solve this equation for nonhomogeneous terms of increasing complexity.

(Case 0) $\underline{F(t) = 0}$: $mx'' + kx = 0$

characteristic equation $mr^2 + k = 0$

$$r = \pm\sqrt{\frac{-k}{m}} = \pm i\sqrt{\frac{k}{m}}$$

general solution $x(t) = c_1 \cos\sqrt{\frac{k}{m}}\,t + c_2 \sin\sqrt{\frac{k}{m}}\,t$

This reveals the physical meaning of $\sqrt{\frac{k}{m}}$, it is the

Natural angular frequency: $\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$

(Case 1) $\underline{F(t) = F_0 \cos\omega t}$: We use undetermined coefficients to find a periodic solution.

Try $x(t) = A\cos\omega t$

$$\begin{aligned} mx'' + kx &= -m\omega^2 A\cos\omega t + kA\cos\omega t \\ &= (k - m\omega^2)A\cos\omega t \end{aligned}$$

We want this to equal $F(t) = F_0\cos\omega t$,

So we need

$$(k - m\omega^2)A = F_0$$

$$A = \frac{F_0}{k - m\omega^2} \Rightarrow x(t) = \frac{F_0}{k - m\omega^2}\cos\omega t$$

We can write $\dfrac{F_0}{k-m\omega^2} = \dfrac{F_0/m}{(k-m\omega^2)/m} = \dfrac{F_0/m}{\frac{k}{m}-\omega^2} = \dfrac{F_0/m}{\omega_0^2-\omega^2}$

In summary, a periodic solution of $mx''+kx = F_0\cos\omega t$ is

$$x(t) = \frac{F_0/m}{\omega_0^2-\omega^2}\cos\omega t$$

This only works if $\omega \neq \omega_0$, that is, the driving frequency is not equal to the natural frequency.
If, conversely, $\omega = \omega_0$, <u>resonance</u> occurs, and there is no periodic solution.

(Case 2) $F(t) = F_1\cos\omega_1 t + F_2\cos\omega_2 t$

For this case we will use the previous case plus a version of the <u>principle of superposition</u>:

Suppose $x_1(t)$ satisfies $mx_1''+kx_1 = F_1(t)$
and $x_2(t)$ satisfies $mx_2''+kx_2 = F_2(t)$
Then $x(t) = x_1(t)+x_2(t)$ satisfies

$$mx''+kx = F_1(t)+F_2(t)$$

<u>Proof</u>:
$$\begin{aligned} mx''+kx &= m(x_1+x_2)''+k(x_1+x_2) \\ &= mx_1''+mx_2''+kx_1+kx_2 \\ &= \underbrace{mx_1''+kx_1}_{F_1(t)} + \underbrace{mx_2''+kx_2}_{F_2(t)} \\ &= F_1(t)+F_2(t) \end{aligned}$$

This holds more generally for any linear differential equation.

suppose $\begin{Bmatrix} p(D)\, y_1 = f_1 \\ p(D)\, y_2 = f_2 \end{Bmatrix}$ then $p(D)\,(y_1+y_2) = p(D)\,y_1 + p(D)\,y_2 = f_1+f_2$

So, to solve $mx'' + kx = F_1 \cos \omega_1 t + F_2 \cos \omega_2 t$

Solve $mx_1'' + kx_1 = F_1 \cos \omega_1 t$

$mx_2'' + kx_2 = F_2 \cos \omega_2 t$

$$x_1(t) = \frac{F_1/m}{\omega_0^2 - \omega_1^2} \cos \omega_1 t$$

$$x_2(t) = \frac{F_2/m}{\omega_0^2 - \omega_2^2} \cos \omega_2 t$$

So $x(t) = x_1(t) + x_2(t) = \dfrac{F_1/m}{\omega_0^2 - \omega_1^2} \cos \omega_1 t + \dfrac{F_2/m}{\omega_0^2 - \omega_2^2} \cos \omega_2 t$

is a particular solution of the original equation.

Eg. Solve $x'' + x = \cos 2t + \cos 3t$

Here, $m = 1$, $k = 1$, $\omega_0 = \sqrt{\frac{k}{m}} = 1$

Solve $x_1'' + x_1 = \cos 2t \Longrightarrow x_1(t) = \dfrac{1}{1^2 - 2^2} \cos 2t = \dfrac{-1}{3} \cos 2t$

$x_2'' + x_2 = \cos 3t \Longrightarrow x_2(t) = \dfrac{1}{1^2 - 3^2} \cos 3t = \dfrac{-1}{8} \cos 3t$

So $x(t) = \dfrac{-1}{3} \cos 2t - \dfrac{1}{8} \cos 3t$ solves original eqn.

Expanding on case 2, we can do $mx''+kx=F(t)$ where $F(t)$ is a sum of cosines.

Using the solutn of $mx''+kx = F_0 \sin \omega t$ which is

$$x(t) = \frac{F_0/m}{\omega_0^2-\omega^2} \sin \omega t$$

We can also do it if $F(t)$ is a sum of sines and cosines

But the point of Fourier series is that (essentially) <u>any</u> periodic $F(t)$ is a sum of sines and cosines

<u>Case 3</u>: $F(t) = \sum_{n=1}^{\infty} a_n \cos n\Omega t$ a Fourier cosine series

Here Ω is the fundamental angular frequency of $F(t)$
The fundamental period is $\frac{2\pi}{\Omega} = 2L$

so $L = \frac{\pi}{\Omega}$ or $\Omega = \frac{\pi}{L}$

Solve term-by-term $mx_n'' + kx_n = a_n \cos n\Omega t$

$\omega_0 = \sqrt{\frac{k}{m}}$, $F_0 = a_n$, $\omega = n\Omega$, so $x_n(t) = \frac{a_n/m}{\omega_0^2-(n\Omega)^2} \cos n\Omega t$

A solution of $mx''+kx = F(t)$ is then

$$x(t) = \sum_{n=1}^{\infty} x_n(t) = \sum_{n=1}^{\infty} \frac{a_n/m}{\omega_0^2-(n\Omega)^2} \cos n\Omega t$$

(which is a Fourier cosine series for $x(t)$)

This solution method will be valid as long as none of the denominators $\omega_0^2-(n\Omega)^2$ are zero. Otherwise, resonance occurs, and there is no periodic solution.

Resonance: $\omega_0^2-(n\Omega)^2=0 \iff \omega_0^2=(n\Omega)^2$

$$\iff \omega_0 = n\Omega$$

$$\iff \frac{\omega_0}{\Omega} = n$$

So resonance can occur only if $\frac{\omega_0}{\Omega}$ is an integer.

That is, only if the natural frequency is an integer multiple of the fundamental frequency of the driving force.