Convergence of Fourier Series

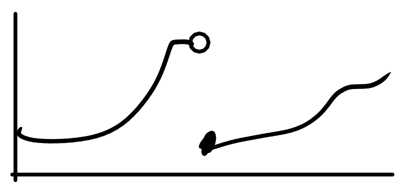
A function $f(t)$ is called <u>smooth</u> if

- $f(t)$ is continuous
- $f'(t)$ exist for all t
- $f'(t)$ is continuous

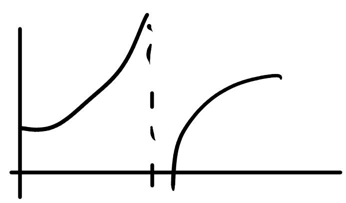
<u>Thm</u> If $f(t)$ is a smooth periodic function, the Fourier series of $f(t)$ converges to $f(t)$ for every t.

A function is <u>piecewise smooth</u> if it is smooth except for (isolated) "jump discontinuities"

Jump discontinuity



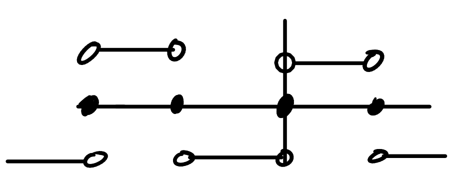
Vertical asymptote is <u>not</u> a jump discontinuity.



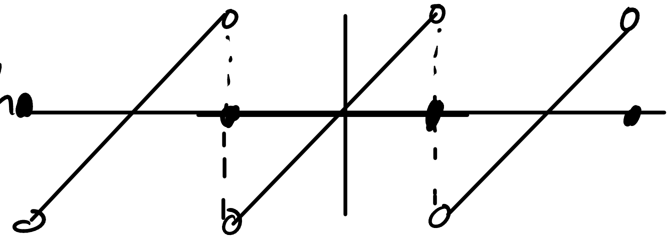
<u>Thm</u> If $f(t)$ is a piecewise smooth periodic function, the Fourier series of f converges to

- $f(t)$ if f is continuous at t
- $\frac{1}{2}\left[\lim_{s \to t^-} f(s) + \lim_{s \to t^+} f(s)\right]$ if t is a jump discontinuity.

Eg. Squarewave Fourier series converges to



For sawtooth



Differentiation of Fourier Series.

Suppose $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$

If we differentiate term by term, we expect

$$f'(t) \overset{?}{=} 0 + \sum_{n=1}^{\infty}\left(-\frac{n\pi}{L} a_n \sin\frac{n\pi t}{L} + \frac{n\pi}{L} b_n \cos\frac{n\pi t}{L}\right)$$

This is <u>not always valid</u> (see HW)

It is valid as long as
- $f(t)$ is continuous and
- $f'(t)$ is piecewise smooth.

Integration works better:
Suppose $f(t)$ is piecewise continuous, and it has Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$$

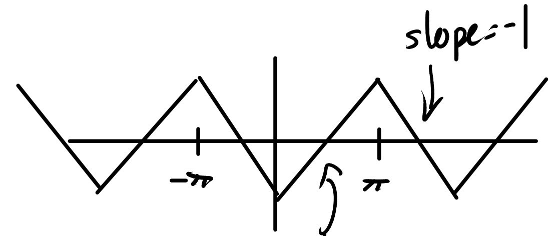
* It is possible that the Fourier series of a merely piecewise continuous function does not converge at all. *

Even so, the equation

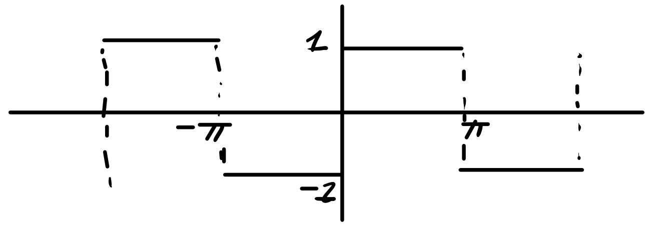
$$\int_0^t f(s)\,ds = \frac{a_0 t}{2} + \sum_{n=1}^{\infty}\left[\frac{L}{n\pi} a_n \sin\frac{n\pi t}{L} - \frac{L}{n\pi} b_n\left(\cos\frac{n\pi t}{L} - 1\right)\right]$$

is valid (in particular, the sum is convergent)

<u>Example</u> Triangle wave $f(t) =$



$f'(t) =$



Square wave!

We know Fourier series of $f'(t)$

$$f'(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin nt$$

What is the Fourier series of $f(t)$?

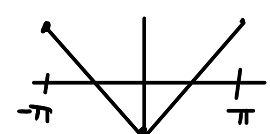
$$f(t) - f(0) = \int_0^t f'(s)\,ds = \int_0^t \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin ns \,ds$$

$$= \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \int_0^t \sin ns\,ds = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \left[\frac{-1}{n} \cos ns\right]_{s=0}^{s=t}$$

$$= \frac{4}{\pi} \sum_{n \text{ odd}} \frac{-1}{n^2} (\cos nt - 1) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} \cos nt$$

So $$f(t) = \underbrace{\left(f(0) + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2}\right)}_{\text{just a constant.}} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} \cos nt$$

Suppose $f(t)$ is symmetrical about the t-axis



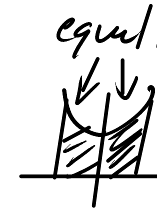
Then $a_0 = 0$, and $f(0) = -\frac{\pi}{2}$.

This implies $\displaystyle\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$!

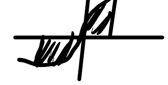
Recall $F(t)$ is <u>even</u> if $F(-t) = F(t)$

<u>odd</u> if $F(-t) = -F(t)$

If $F(t)$ is <u>even</u> then $\int_{-a}^{a} F(t)\,dt = 2\int_0^a F(t)\,dt$



If $F(t)$ is <u>odd</u> then $\int_{-a}^{a} F(t)\,dt = 0$



areas cancel

<u>If $f(t)$ is even</u>, then $f(t)\cos\left(\frac{n\pi t}{L}\right)$ is even,

and $f(t)\sin\left(\frac{n\pi t}{L}\right)$ is odd.

look $f(-t)\cos\left(\frac{n\pi(-t)}{L}\right) = f(t)\cos\frac{n\pi t}{L}$

$f(-t)\sin\left(\frac{n\pi(-t)}{L}\right) = f(t)\left(-\sin\frac{n\pi t}{L}\right) = -f(t)\sin\frac{n\pi t}{L}$

So $a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{2}{L}\int_0^L f(t)\,dt$

$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt$

$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt = 0$

<u>If $f(t)$ is odd</u>; then $f(t)\cos\frac{n\pi t}{L}$ is odd

and $f(t)\sin\frac{n\pi t}{L}$ is even

So $a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = 0$

$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt = 0$

$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt = \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi t}{L}\,dt$