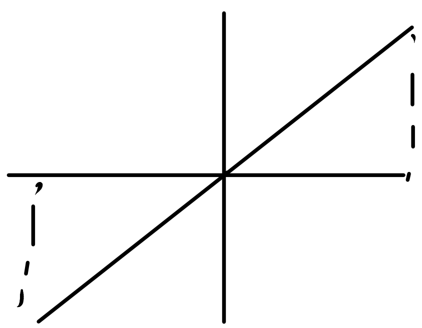
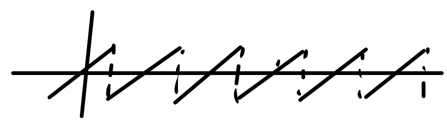
Fourier Series; Pointwise convergence.

Another example of Fourier Series: Sawtooth wave



$$f(t) = \begin{cases} t & \text{if } -1 \le t < 1, \\ \text{repeats periodically with} \\ \text{period } 2 \ (L = 1). \end{cases}$$



What is $f(27.2)$?

$$f(27.2) = f(25.2) = f(23.2) = \cdots f(1.2) = f(-.8) = -.8$$

In general, for a function with period $2L$: $f(t+2Lk) = f(t)$

Compute Fourier coefficients

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(t)\,dt = \int_{-1}^{1} t\,dt = \left[\frac{1}{2}t^2\right]_{-1}^{1} = \frac{1}{2}(1-1) = 0$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos\frac{n\pi t}{1}\,dt = \int_{-1}^{1} t\cos n\pi t\,dt$$

Parts $\quad u = t \quad dv = \cos n\pi t\,dt$

$\qquad\quad du = dt \quad v = \frac{1}{n\pi}\sin n\pi t$

$$\int t\cos n\pi t\,dt = \frac{1}{n\pi} t\sin n\pi t - \int \frac{1}{n\pi}\sin n\pi t\,dt$$

$$= \frac{1}{n\pi} t\sin n\pi t + \left(\frac{1}{n\pi}\right)^2 \cos n\pi t + C$$

$$a_n = \left[\frac{1}{n\pi} t \sin n\pi t + \left(\frac{1}{n\pi}\right)^2 \cos n\pi t \right]_{-1}^{1}$$

$$= \frac{1}{n\pi} \sin n\pi + \frac{1}{n\pi} \sin(-n\pi) + \left(\frac{1}{n\pi}\right)^2 \cos n\pi - \left(\frac{1}{n\pi}\right)^2 \cos(-n\pi)$$

$$= 0 + 0 + \left(\frac{1}{n\pi}\right)^2 - \left(\frac{1}{n\pi}\right)^2 = 0$$

So $a_0 = 0$ and $a_n = 0$.

We could have seen this because $f(t)$, $f(t)\cos n\pi t$ are odd functions.

- If $F(t)$ is odd: $F(-t) = -F(t)$ Then $\int_{-a}^{a} F(t)\,dt = 0$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(t) \sin n\pi t \, dt = \int_{-1}^{1} t \sin n\pi t \, dt$$

Parts: $\int t \sin n\pi t \, dt = -\frac{1}{n\pi} t \cos n\pi t - \int -\frac{1}{n\pi} \cos n\pi t \, dt$

$$\boxed{\begin{array}{ll} u = t & dv = \sin n\pi t \, dt \\ du = dt & v = -\frac{1}{n\pi}\cos n\pi t \end{array}}$$

$$= -\frac{1}{n\pi} t \cos n\pi t + \left(\frac{1}{n\pi}\right)^2 \sin n\pi t + C$$

$$b_n = \left[-\frac{1}{n\pi} t \cos n\pi t + \left(\frac{1}{n\pi}\right)^2 \sin n\pi t \right]_{-1}^{1}$$

$$= -\frac{1}{n\pi} \cos n\pi + \frac{1}{n\pi}(-1)\cos(-n\pi) + \left(\frac{1}{n\pi}\right)^2 \sin n\pi - \left(\frac{1}{n\pi}\right)^2 \sin(-n\pi)$$

$$= -\frac{1}{n\pi}\left(\cos n\pi + \cos n\pi\right) + 0 - 0$$

$$= -\frac{2}{n\pi}\cos n\pi = \begin{cases} \frac{2}{n\pi} & n \text{ odd} \\ -\frac{2}{n\pi} & n \text{ even} \end{cases}$$

We can also write $\cos n\pi = (-1)^n$

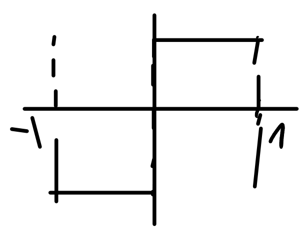
so $b_n = \frac{-2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}$

So the Fourier series for the sawtooth is

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin n\pi t$$

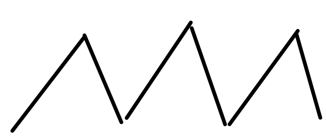
Compare with square wave of same period

$$\sum_{n \text{ odd}} \frac{4}{n\pi} \sin n\pi t$$



In the squarewave, <u>only odd</u> multiples of the fundamental occur
In the sawtooth, <u>all</u> multiples of the fundamental occur.
In <u>both</u>, the amplitude of the nth multiple is proportional to $1/n$.

For triangle wave



Odd multiples of fundamental are present, but the nth has amplitude proportional to $1/n^2$.

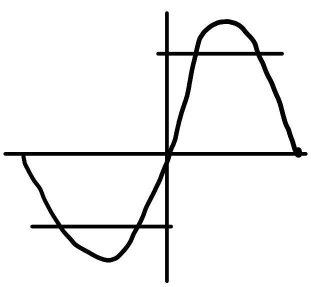
Convergence: Do the partial sums of the Fourier Series converge to the original function?

If the function is "smooth" meaning $f(t)$ is continuous, $f'(t)$ exists everywhere, and $f'(t)$ is continuous, then the answer is <u>yes</u>.

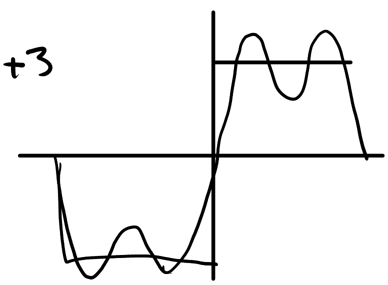
If $f(t)$ is discontinuous, the answer is <u>not necessarily</u>.

Partial sums of Fourier series for square wave

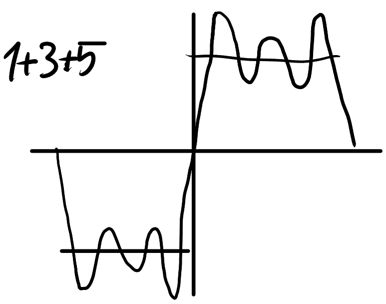
1



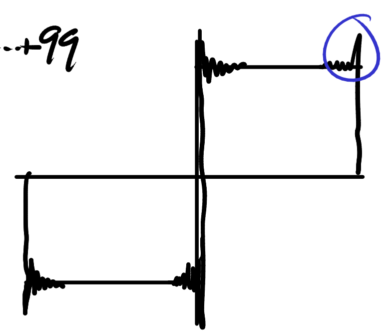
1+3



1+3+5



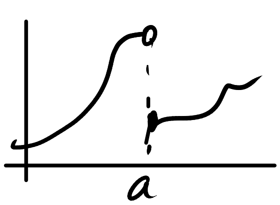
1+3+...+99



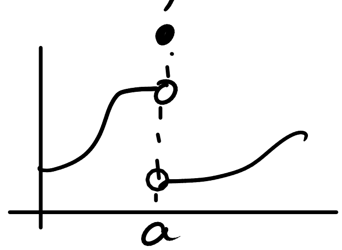
"Gibbs Phenomenon"

Basic issue: sequence of continuous functions can't converge (uniformly) to a discontinuous function!

A function is said to have a <u>jump discontinuity</u> if both one-sided limits exist, but are not equal.



or



$$\lim_{t \to a^-} f(t) \neq \lim_{t \to a^+} f(t)$$

In this situation the Fourier Series converges to the average of the two one sided limits!

$$\text{FourierSeries}(a) = \frac{1}{2}\left[\lim_{t\to a^-} f(t) + \lim_{t\to a^+} f(t)\right]$$

Eg. For square wave: