Fourier Coefficients:

Recall orthogonality relations

Fundamental period 2π

$\sin nt$, $\cos nt$, $n=1,2,3,\ldots$

$$\int_{-\pi}^{\pi} \cos mt \cos nt \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \sin nt \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \cos nt \, dt = 0$$

Fundamental period $2L$

$\sin \frac{n\pi t}{L}$, $\cos \frac{n\pi t}{L}$, $n=1,2,3,\ldots$

$$\int_{-L}^{L} \cos \frac{m\pi t}{L} \cos \frac{n\pi t}{L} \, dt = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

$$\int_{-L}^{L} \sin \frac{m\pi t}{L} \sin \frac{n\pi t}{L} \, dt = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin \frac{m\pi t}{L} \cos \frac{n\pi t}{L} \, dt = 0$$

Now, suppose we are trying to write a function $f(t)$ (which we assume is periodic with period $2L$) as a sum of sines and cosines with fundamental period $2L$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right)$$

To figure out constant term, just integrate from $-L$ to L

$$\int_{-L}^{L} f(t)\,dt = \int_{-L}^{L}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right) dt$$

look: $$\int_{-L}^{L} \cos\frac{n\pi t}{L}\,dt = \left[\frac{L}{\pi n}\sin\frac{n\pi t}{L}\right]_{-L}^{L} = \frac{L}{\pi n}\left[\sin n\pi - \sin -n\pi\right]$$

$$= \frac{L}{\pi n}\cdot 0 = 0$$

$$\int_{-L}^{L} \sin\frac{n\pi t}{L}\,dt = \left[\frac{-L}{\pi n}\cos\frac{n\pi t}{L}\right]_{-L}^{L} = -\frac{L}{\pi n}\left[\cos n\pi - \cos(-n\pi)\right]$$

$$= -\frac{L}{\pi n}\cdot 0 = 0$$

So all of the integral goes away. except the $\frac{a_0}{2}$ term

$$\int_{-L}^{L} f(t)\,dt = \int_{-L}^{L}\frac{a_0}{2}\,dt = \frac{a_0}{2}\int_{-L}^{L} dt = \frac{a_0}{2}2L = a_0 L$$

$$\int_{-L}^{L} f(t)\,dt = a_0 L \implies a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

So we found a_0 in terms of $f(t)$!

To find a_m: multiply by $\cos\frac{m\pi t}{L}$ and integrate:
use orthogonality

$$\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt = \int_{-L}^{L}\left(\frac{a_0}{2} + \sum a_n\cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)\cos\frac{m\pi t}{L}\,dt$$

$$= \int_{-L}^{L} \frac{a_0}{2}\cos\frac{m\pi t}{L} + \sum \quad a_n \cos\frac{n\pi t}{L}\cos\frac{m\pi t}{L} + b_n \sin\frac{n\pi t}{L}\cos\frac{m\pi t}{L}\Big)\,dt$$

By orthogonality, all the terms go away except

$$\int_{-L}^{L} a_n \cos\frac{n\pi t}{L}\cos\frac{m\pi t}{L}\,dt \quad \text{when} \quad m=n$$

$$= a_m L$$

So $\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}\,dt$ so $a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}\,dt$

Similar argument shows $b_m = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{m\pi t}{L}\,dt$

Definition The Fourier coefficients of $f(t)$ are

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \quad n=1,2,\dots$$

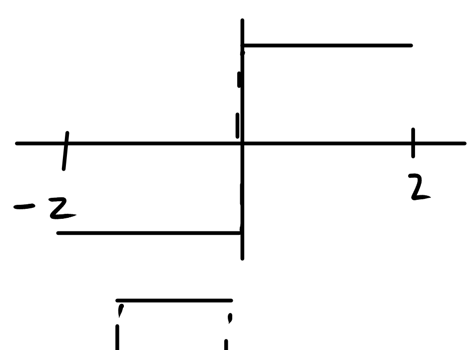
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt \quad n=1,2,\dots$$

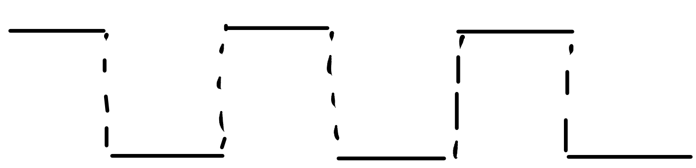
The Fourier Series of $f(t)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{L} + b_n\sin\frac{n\pi t}{L}\right)$$

Examples: square-wave with period $4 = 2L$, $L=2$.

$$f(t) = \begin{cases} 1 & 0 \le t < 2 \\ -1 & -2 \le t < 0 \end{cases}$$



Repeats periodically 

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(t)\,dt = \frac{1}{2}\left[\int_{-2}^{0} -1\,dt + \int_{0}^{2} 1\,dt\right] = \frac{1}{2}\Big[(-2)+(2)\Big] = 0$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(t)\cos\frac{n\pi t}{2}\,dt = \frac{1}{2}\left[\int_{-2}^{0} -\cos\frac{n\pi t}{2}\,dt + \int_{0}^{2}\cos\frac{n\pi t}{2}\,dt\right]$$

$$= \frac{1}{2}\left\{\left[-\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right]_{-2}^{0} + \left[\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right]_{0}^{2}\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{-2}{n\pi}\right)\big(\sin 0 - \sin(-n\pi)\big) + \left(\frac{2}{n\pi}\right)\big(\sin n\pi - \sin 0\big)\right\}$$

$$= 0$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(t)\sin\frac{n\pi t}{2}\,dt = \frac{1}{2}\left\{\int_{-2}^{0} -\sin\frac{n\pi t}{2}\,dt + \int_{0}^{2}\sin\frac{n\pi t}{2}\,dt\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{2}{n\pi}\cos\frac{n\pi t}{2}\right]_{-2}^{0} + \left[-\frac{2}{n\pi}\cos\frac{n\pi t}{2}\right]_{0}^{2}\right\}$$

$$= \frac{1}{2}\cdot\frac{2}{n\pi}\left\{\cos 0 - \cos(-n\pi) - \cos(n\pi) + \cos 0\right\}$$

$$= \frac{1}{2}\frac{2}{n\pi}\left\{2\cos 0 - 2\cos(n\pi)\right\}$$

$$= \frac{2}{n\pi}\left(1 - \cos n\pi\right) \qquad \cos n\pi = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right) = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \qquad \cos n\pi = (-1)^n$$

So Fourier series is

$$\sum_{n \text{ odd}} \frac{4}{n\pi} \sin \frac{n\pi t}{2} = \frac{4}{\pi}\left(\sin \frac{\pi t}{2} + \frac{1}{3} \sin \frac{3\pi t}{2} + \cdots \right)$$