Equations solvable by direct integration

Last time: Newton's Law of cooling

$$\frac{dT}{dt} = -k(T-A)$$

How to solve? One asks, why not just integrate with respect to t?

$$T = \int \frac{dT}{dt}\,dt = \underbrace{\int -k\,(T-A)\,dt}$$

This is true ... but here we are integrating the unknown function $T(t)$, so we can't do this integral. FAIL

We failed because the Right hand side of the DE. depends on the unknown function T

$$\frac{dT}{dt} = -k(T-A)$$

But, this strategy will work if the right hand side doesn't depend on the unknown function.

Let's formulate this case precisely

Notation: x — independent variable
y — dependent variable
$y(x)$ - unknown function we wish to find

Consider D.E. of the form

$$\boxed{\frac{dy}{dx} = f(x)}$$

where $f(x)$ is some given function

Eg $\frac{dy}{dx} = 7x^2 + 4$, $\frac{dy}{dx} = e^{2x}$, $\frac{dy}{dx} = \arctan(x)$

We can (at least theoretically) integrate both sides dx to get solution.

Eg $\frac{dy}{dx} = 7x^2+4$

$$\int \frac{dy}{dx}\, dx = \int (7x^2+4)\, dx$$

Fund. Thm. Calc.

$$y = \frac{7}{3}x^3 + 4x + C$$

Because we are doing indefinite integrals, we need to put a constant of integration in this equation

For the equation

In abstract terms: $\dfrac{dy}{dx} = f(x)$

General solution is $y(x) = \int f(x)\,dx + C$

The constant of integration is not just some pedantic thing any more: We need it in order to satisfy an initial condition

We need the initial condition because the differential equation tells us how y changes, but it doesn't tell us where y starts

Consider problem $\begin{cases} \dfrac{dy}{dx} = 7x^2 + 4 \\ y(0) = 9 \end{cases}$

We showed $y(x) = \frac{7}{3}x^3 + 4x + C$ for some constant C

Plug in $x=0$ $\quad y(0) = \frac{7}{3}0^3 + 4\cdot 0 + C = C$

So set $\boxed{C = 9}$

$y(x) = \frac{7}{3}x^3 + 4x + 9$ solves $\begin{cases} \dfrac{dy}{dx} = 7x^2 + 4 \\ y(0) = 9 \end{cases}$

This is called a particular solution

Try another $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$ $\quad y(0) = \pi$

$$y(x) = \int \frac{1}{1+x^2}\,dx = \arctan(x) + C \leftarrow \text{General solution}$$

$$\pi = y(0) = \arctan(0) + C = 0 + C = C$$

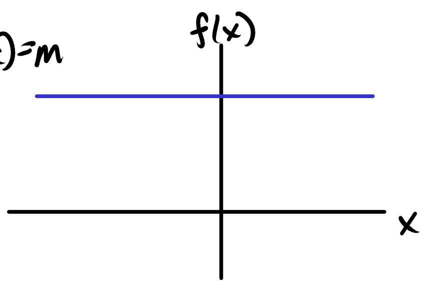
so $y(x) = \arctan(x) + \pi$ is the particular solution

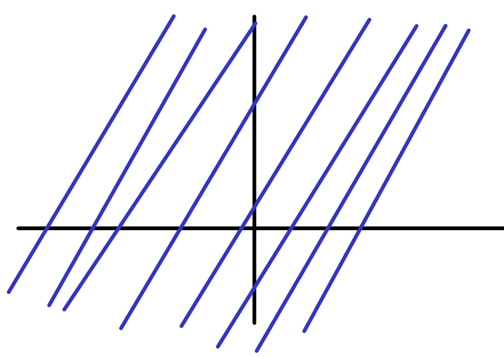
What if you just know $f(x)$ from its graph

$$\frac{dy}{dx} = f(x)$$

$f(x)$	$y(x)$

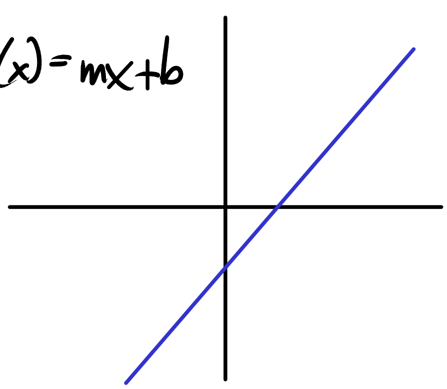
$f(x)=m$

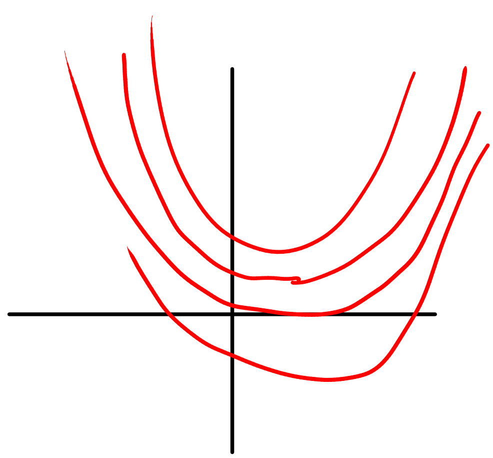




$y = mx + C$
straight lines

$f(x) = mx+b$





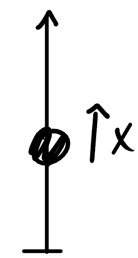
$y(x)$
parabolae

min/max where $f(x) = 0$.

This method extends to higher-order differential equations

Eg $\frac{d^2y}{dx^2} = f(x)$ let's do this in terms of the application to kinematics

Notation: t – time
x – position of particle on a line
v – velocity
a – acceleration



Definitions $v = \frac{dx}{dt}$, $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Problem: prescribed acceleration

$a(t) = f(t)$, where $f(t)$ is some fixed function

That is, $\frac{d^2x}{dt^2} = f(t)$

[By Newtons 2nd Law $f(t) = \frac{\text{Force}(t)}{\text{mass}}$
So prescribed acceleration is equivalent to prescribed Force.]

Solution integrate twice!

First find $v(t)$ from $\frac{dv}{dt} = a(t) = f(t)$

$$v(t) = \int f(t)\,dt + C$$

constant of integration

Then find $x(t)$ from $\frac{dx}{dt} = v(t)$, which we now know

$$x(t) = \int v(t)\,dt + D$$

another constant of integration

All told, our solution has <u>two</u> constants of integration
We will need two initial conditions as well

<u>Eg</u> constant acceleration $a(t) = -g$ (downward gravity)

$$v(t) = \int a(t)\,dt = \int -g\,dt = -gt + C$$

$$x(t) = \int v(t)\,dt = \int (-gt + C)\,dt = -\frac{1}{2}gt^2 + Ct + D$$

two constants

General solution $x(t) = -\frac{1}{2}gt^2 + Ct + D$

Now $x(0) = D$ so D represents initial position

And $v(0) = C$ so C represents initial velocity

If we specify the initial position and velocity, we will get a particular solution.