Nonhomogeneous equations II – The resonant case

Sometimes the method of undetermined coefficients will fail.

For example, Try solving $(D-2)y = e^{2x}$ or $y' - 2y = e^{2x}$

Based on what we said last time, we should try Ae^{2x}

$$(D-2)(Ae^{2x}) = 2Ae^{2x} - 2Ae^{2x} = 0 \stackrel{?}{=} e^{2x}$$

It can't work!

The problem is that the nonhomogeneous term e^{2x} has something in common with the solutions of the homogeneous equation $(D-2)y = 0$!

This is called (mathematical) <u>resonance</u>.

Another example: $(D^2+D)y = x$

$y = Ax + B$ doesn't work, but $y = Ax^2 + Bx + C$ does

$$(D^2+D)(Ax^2+Bx+C) = 2A + 2Ax + B$$
$$= 2Ax + (2A+B)$$

want $= x$

So $2A = 1$ $\quad A = 1/2$ $\quad y_p = \frac{1}{2}x^2 - x$

$2A + B = 0$ $\quad B = -1$

$C =$ anything!

General slogan: Need to include higher power of x times the functions you already have

Another example $(D^2+1)y = \sin(x)$

Since $\sin(x)$ and $\cos(x)$ both solve $(D^2+1)y=0$,
we need to include $x\sin(x)$ and $x\cos(x)$

$$(D^2+1)\left(Ax\sin(x) + Bx\cos(x) + C\sin(x) + D\cos(x)\right)$$

$$D^2(x\sin x) = D(x\cos x + \sin x) = -x\sin x + \cos x + \cos x$$
$$(D^2+1)(x\sin x) = 2\cos x$$
$$D^2(x\cos x) = D(-x\sin x + \cos x) = -x\cos x - \sin x - \sin x$$
$$(D^2+1)(x\cos x) = -2\sin x$$

$$\to \quad A\cdot 2\cos x - B\cdot 2\sin x + 0 + 0$$

want $= \sin x$

So let $A = 0$, $B = -\frac{1}{2}$

$$\boxed{y_p = -\frac{1}{2}x\cos(x)}$$

Why does this work? How high of a power of x must you take?
Answers in terms of a more systematic method of solving nonhomogeneous equations.

The Annihilator method

Consider a function $f(x)$. An <u>annihilator</u> of $f(x)$ is a differential operator $A(D)$ such that

$$A(D)(f(x)) = 0$$

$f(x)$	$A(D)$
x^3	D^4
e^{ax}	$D-a$
x^2e^{ax}	$(D-a)^3$
$\sin(3x)$	D^2+9
$x\sin(3x)$	$(D^2+9)^2$
$e^{ax}+e^{bx}$	$(D-a)(D-b)$
$xe^{ax}+x^2e^{bx}$	$(D-a)^2(D-b)^3$
$\vdots$	$\vdots$

To solve
$$p(D)\,y = f(x)$$
Find annihilator $A(D)$ so that $A(D)\cdot f(x)=0$ (if possible)

Then apply $A(D)$ to both sides
$$p(D)\,y = f(x)$$
$$A(D)\,p(D)\,y = A(D)\,f(x) = 0$$

So the function y we want also solves the homogeneous equation
$$A(D)\,p(D)\,y = 0.$$

Find the general solution of this equation, and try that in the nonhomogeneous equation.

Example $(D-5)\,y = xe^{5x}$

$f(x) = xe^{5x}$ annihilator $= (D-5)^2$

$$(D-5)^2(D-5)\,y = (D-5)^2(xe^{5x}) = 0$$
$$(D-5)^3 y = 0$$

So $y = Ae^{5x} + Bxe^{5x} + Cx^2e^{5x}$

Try it in the original equation

$$(D-5)(Ae^{5x}+Bxe^{5x}+Cx^2e^{5x})$$

$$= 0 + Be^{5x} + 2Cxe^{5x}$$

want $= xe^{5x}$

so $B=0$ and $2C=1$ or $C=\frac{1}{2}$ A = anything

Thus $y_p = \frac{1}{2}x^2e^{5x}$ is a <u>particular solution</u>

In fact $y = Ae^{5x}+\frac{1}{2}x^2e^{5x}$ is the <u>general solution</u>

* The annihilator method gives you the general solution.

Another example $(D-1)y = e^x + \sin(x)$

e^x annihilator $(D-1)$
$\sin(x)$ annihilator (D^2+1)
$e^x+\sin(x)$ annihilator $(D-1)(D^2+1)$

$$(D-1)(D^2+1)(D-1)y = (D-1)(D^2+1)[e^x+\sin(x)] = 0$$

$$(D^2+1)(D-1)^2 y = 0$$

$$y = \underbrace{A\sin(x)+B\cos(x)}_{\text{from } D^2+1} + \underbrace{Ce^x+Exe^x}_{\text{from } (D-1)^2}$$

Try in $\quad (D-1)\,y = e^x + \sin(x)$

$$(D-1)\left[A\sin(x) + B\cos(x) + Ce^x + Exe^x\right]$$

$$= A\cos x - A\sin x - B\sin x - B\cos x + Ce^x - Ce^x$$
$$+ E(xe^x + e^x) - Exe^x$$

$$= (A-B)\cos x - (A+B)\sin x + Ee^x$$

want $= e^x + \sin x$

$$A - B = 0 \qquad -(A+B) = 1 \qquad E = 1$$

$$A = B \qquad -2A = 1$$

$$A = -\frac{1}{2} \qquad B = -\frac{1}{2}$$

$$y = -\frac{1}{2}\sin x - \frac{1}{2}\cos x + Ce^x + xe^x$$

This is the general solution! it has one arbitrary constant, C.