Solving Nonhomogeneous equations I

Undetermined coefficients

This technique grows out of the "trial-and-error method"

Find a particular solution of $y''+y'+2y=\sin(x)$

$p(D)=D^2+D+2$ Want to solve $p(D)y=\sin(x)$

Why not try a multiple of $\sin(x)$, $A\sin(x)$?

$$\begin{aligned}
p(D)(A\sin(x)) &= A\sin''(x)+A\sin'(x)+2A\sin(x)\\
&= -A\sin(x)+A\cos(x)+2A\sin(x)\\
&= A\sin(x)+A\cos(x)
\end{aligned}$$

If we take $A=1$, we get $\sin(x)$, but also $\cos(x)$.
So why not include a $\cos(x)$ term as well?

$$p(D)(A\sin(x)+B\cos(x))=(D^2+D+2)(A\sin(x)+B\cos(x))$$

$$= -A\sin(x)-B\cos(x)+A\cos(x)-B\sin(x)+2A\sin(x)+2B\cos(x)$$

$$= (-A-B+2A)\sin(x)+(-B+A+2B)\cos(x)$$

$$= (A-B)\sin(x)+(A+B)\cos(x)$$

Want this $=\sin(x)$, so want $\begin{matrix} A-B=1 \\ A+B=0 \end{matrix}$ $\quad \begin{matrix} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{matrix}$

Thus $y_p = \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)$ is a particular solution.

The general solution is gotten by adding to this the general solution of the homogeneous equation $(D^2+D+2)y=0$

$$r^2+r+2=0 \qquad r = \frac{-1\pm\sqrt{1-8}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{7}}{2}$$

solutions $y_c = e^{-\frac{1}{2}x}\left(c_1\cos\left(\frac{\sqrt{7}}{2}x\right) + c_2\sin\left(\frac{\sqrt{7}}{2}x\right)\right)$

General solution of $(D^2+D+2)y = \sin(x)$ is

$$y = \underbrace{e^{-\frac{1}{2}x}\left(c_1\cos\left(\frac{\sqrt{7}}{2}x\right) + c_2\sin\left(\frac{\sqrt{7}}{2}x\right)\right)}_{y_c} + \underbrace{\frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)}_{y_p}$$

$$p(D)(y_c+y_p) = p(D)y_c + p(D)y_p = 0 + \sin(x) \quad \checkmark$$

We can make this "trial and error" method more systematic, but the question is
"What am I supposed to try?"

Somewhat imprecise but useful suggestion:
"Try a linear combination of all the terms that appear in the derivatives of the nonhomogeneous term."

Let's consider $p(D) = D^2+D+2$ again, but now we want to solve

$$p(D)y = x^3$$

Nonhomogeneous term $= x^3$
first derivative $3x^2$
second " $6x$
third " 6
fourth " 0

So try $Ax^3 + Bx^2 + Cx + E$

$$(D^2+D+2)(Ax^3+Bx^2+Cx+E)$$
$$= 6Ax + 2B + 3Ax^2 + 2Bx + C + 2Ax^3 + 2Bx^2 + 2Cx + 2E$$
$$= 2Ax^3 + (3A+2B)x^2 + (6A+2B+2C)x + (2B+C+2E)$$

$2A = 1$ $\qquad A = \frac{1}{2}$
$3A + 2B = 0$ $\qquad \frac{3}{2} + 2B = 0$ $\qquad B = -\frac{3}{4}$
$6A + 2B + 2C = 0$ $\qquad 3 - \frac{3}{2} + 2C = 0$ $\qquad \frac{3}{2} + 2C = 0$ $\qquad C = \frac{-3}{4}$
$2B + C + 2E = 0$ $\qquad -\frac{3}{2} - \frac{3}{4} + 2E = 0$ $\qquad -\frac{9}{4} + 2E = 0$ $\qquad E = \frac{9}{8}$

$$\boxed{y_p = \frac{1}{2}x^3 - \frac{3}{4}x^2 - \frac{3}{4}x + \frac{9}{8}}$$ is a particular solution

How about $(D^2+D+2)\,y = e^{4x}$

Nonhomogeneous term $= e^{4x}$
First derivative $= 4e^{4x}$
Second " $= 16e^{4x}$
Third " $= 64e^{4x}$
⋮

So just try $y_p = Ae^{4x}$

$$(D^2+D+2)Ae^{4x} = 16Ae^{4x} + 4Ae^{4x} + 2Ae^{4x} = 22Ae^{4x}$$

Want $22A = 1$ so $A = \frac{1}{22}$ $\qquad \boxed{y_p = \frac{1}{22}e^{4x}}$

How about $(D^2+D+2)\,y = x^2e^x$

Nonhomogeneous term $= x^2e^x$
First deriv $= x^2e^x + 2xe^x$
Second deriv $= x^2e^x + 2xe^x + 2xe^x + 2e^x$
Third deriv = gonna get stuff like x^2e^x, xe^x and e^x.

So try $\quad y_p = Ax^2e^x + Bxe^x + Ce^x$

Want $\quad (D^2+D+2)(Ax^2e^x + Bxe^x + Ce^x) = x^2e^x$

$$= A(x^2e^x + 4xe^x + 2e^x) + B(xe^x + 2e^x) + Ce^x$$

$$+ A(x^2e^x + 2xe^x) + B(xe^x + e^x) + Ce^x$$

$$+ 2Ax^2e^x + 2Bxe^x + 2Ce^x$$

$$= 4Ax^2e^x + (4A + B + 2A + B + 2B)xe^x + (2A + 2B + C + B + C + 2C)e^x$$

$4A = 1 \qquad A = 1/4$

$6A + 4B = 0 \qquad 3A + 2B = 0 \qquad B = -\frac{3}{8}$

$2A + 3B + 4C = 0 \qquad \frac{1}{2} - \frac{9}{8} + 4C = 0 \qquad \frac{4}{8} - \frac{9}{8} + 4C = 0$

$-\frac{5}{8} + 4C = 0$

$C = \frac{5}{32}$

$$\boxed{y_p = \frac{1}{4}x^2e^x - \frac{3}{8}xe^x + \frac{5}{32}e^x}$$