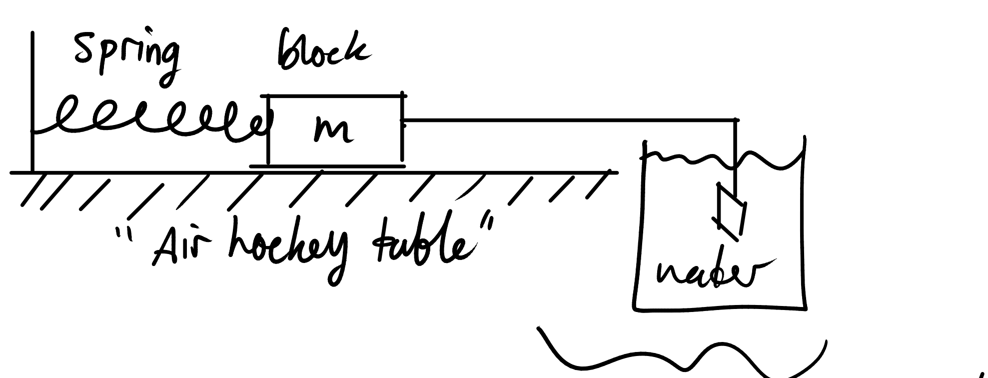
The Damped Oscillator

This is a simple physical system that illustrates all of the mathematical phenomena

<u>The apparatus</u>



This is called a "dashpot"

The block slides left and right, the spring exerts a restoring force, while the dashpot exerts a damping force.

m = mass of block
k = spring constant
c = damping coefficient

x = displacement from equilibrium
spring force $= -kx$
damping force $= -cv = -c\dfrac{dx}{dt}$

Newton says $\quad ma = -kx - cv$

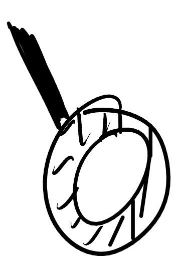
or $\quad ma + cv + kx = 0$

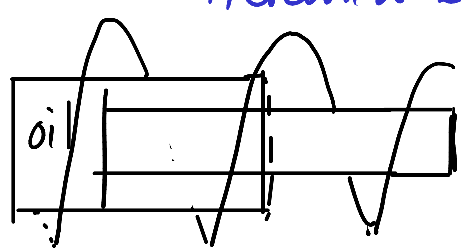
or $\quad m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0$

or $\quad mx'' + cx' + kx = 0$

This is a second-order linear homogeneous constant coefficient differential equation.

Real world Shock absorber





Spring is wrapped around piston

Simplest case: No damping $c=0$

$$mx'' + kx = 0$$

Characteristic equation $\quad mr^2 + k = 0 \qquad r^2 = -\frac{k}{m} \quad r = \pm i\sqrt{\frac{k}{m}}$

Complex roots: A complex solution is $x(t) = e^{i\sqrt{\frac{k}{m}}t}$

Real part: $\cos\left(\sqrt{\frac{k}{m}}\,t\right)$ Imag. part $\sin\left(\sqrt{\frac{k}{m}}\,t\right)$

General solution $x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$

$\sqrt{\frac{k}{m}}$ has units of $(\text{time})^{-1}$ it is called the natural frequency

$$\boxed{\text{Natural frequency: } \omega_0 = \sqrt{\frac{k}{m}}}$$

This is an angular frequency, so the period is $T = \frac{2\pi}{\omega_0}$.

Note that an expression like
$$A\cos(\theta) + B\sin(\theta)$$
is equivalent to one like
$$C\cos(\theta - \alpha) = C(\cos\alpha\cos\theta + \sin\alpha\sin\theta)$$

if we have the correspondence of vectors
$$\langle A, B\rangle = \langle C\cos\alpha, C\sin\alpha\rangle$$
that is if (C, α) is the polar representation of $\langle A, B\rangle$

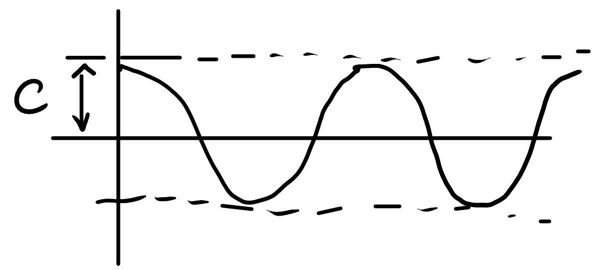
$$C = \sqrt{A^2 + B^2} \qquad \tan\alpha = \frac{B}{A}$$

Thus the general solution can also be written

$$x(t) = C\cos(\omega_0 t - \alpha) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

C = amplitude
α = phase shift

Plot:



Sinusoidal motion

What if damping $c \neq 0$?

$$m x'' + c x' + k x = 0 \qquad r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$
$$m r^2 + c r + k = 0$$

It makes a difference if
$c^2 - 4km < 0$ underdamped
$c^2 - 4km = 0$ critically damped
$c^2 - 4km > 0$ overdamped.

Underdamped $c^2 - 4km < 0$ still have complex roots

$$r = \frac{-c}{2m} \pm i \frac{\sqrt{4km - c^2}}{2m}$$

Abbreviations: $\gamma = \frac{c}{2m} \qquad \omega_0 = \sqrt{\frac{k}{m}}$

Thus $\frac{\sqrt{4km - c^2}}{2m} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \sqrt{\omega_0^2 - \gamma^2}$

$$r = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}$$

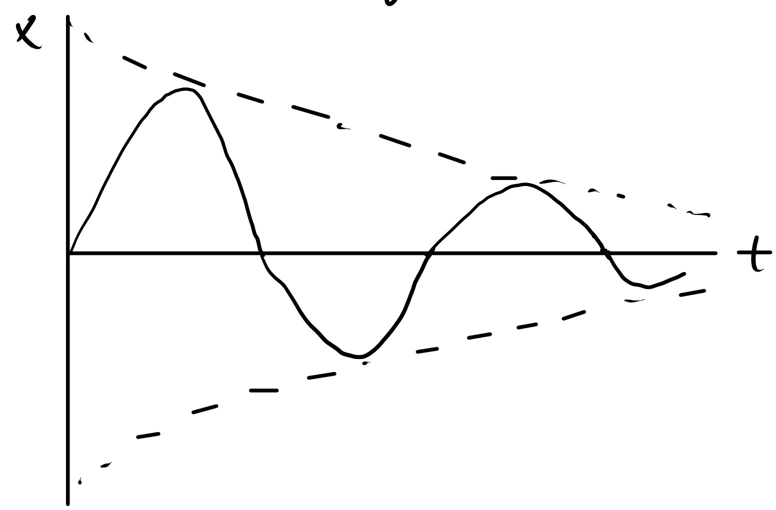
General Solution $x(t) = A\, e^{-\gamma t} \cos\left(\sqrt{\omega_0^2-\gamma^2}\, t\right) + B e^{-\gamma t} \sin\left(\sqrt{\omega_0^2-\gamma^2}\, t\right)$

or $\quad x(t) = C e^{-\gamma t} \cos\left(\sqrt{\omega_0^2-\gamma^2}\, t - \alpha\right)$

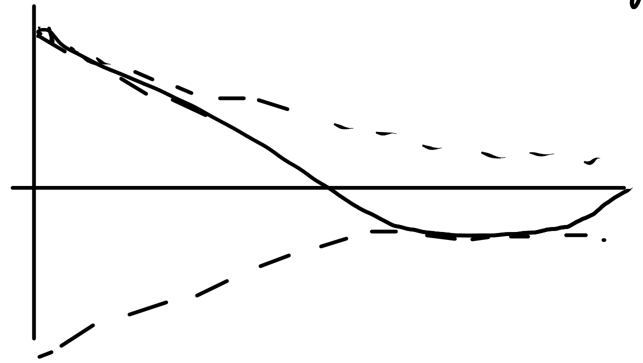
let $\omega = \sqrt{\omega_0^2 - \gamma^2}$ This is the new "frequency" less than the natural frequency.

The "Amplitude" $C e^{-\gamma t}$ decays in time

Plot



As $\gamma = \frac{c}{2m}$ get bigger, the frequency get lower



There is a critical damping coefficient where

$$\omega = \sqrt{\omega_0^2 - \gamma^2} = 0 \qquad \omega_0 = \gamma \quad \sqrt{\frac{k}{m}} = \frac{c}{2m}$$

That is $c^2 = 4km$

<u>Critical damping</u>: $c^2 = 4km$

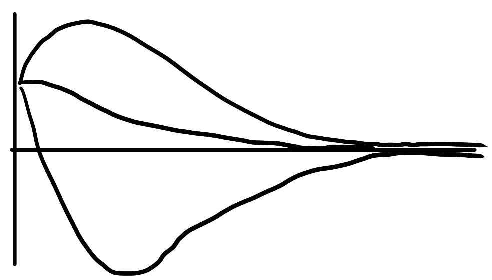
$$mx'' + cx' + kx = 0 \qquad r = \frac{-c + \sqrt{c^2 - 4km}}{2m} = \frac{-c}{2m}$$

$r = -\gamma$ repeated root!

So basic solutions are $e^{-\gamma t}$, $te^{-\gamma t}$

General solution $x(t) = e^{-\gamma t}(c_1 + c_2 t)$

<u>Plot</u>



No actual oscillation!

Overdamped $c^2 > 4km$ $\qquad r = \dfrac{-c \pm \sqrt{c^2 - 4km}}{2m}$

$r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

Basic solutions $e^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t}$, $e^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t}$

General solution $x(t) = e^{-\gamma t}\left(c_1 e^{\sqrt{\gamma^2 - \omega_0^2}\, t} + c_2 e^{-\sqrt{\gamma^2 - \omega_0^2}\, t}\right)$

Could also be written $x(t) = e^{-\gamma t}\left(A\cosh(\sqrt{\gamma^2 - \omega_0^2}\, t) + B\sinh(\sqrt{\gamma^2 - \omega_0^2}\, t)\right)$

<u>Plot</u>