Constant coefficient differential operators.

What is $\frac{d}{dx}$? Well $\frac{d}{dx}[f(x)] = f'(x)$ derivative

The symbol "$\frac{d}{dx}$" represents the operation of differentiation

the operation that takes a function and produces its derivative

$\frac{d}{dx}$ is therefore called the "Derivative operator"

Similarly $\frac{d^2}{dx^2}$ is an operator called the "second derivative operator"

And so on $\frac{d^2}{dx^3}, \ldots, \frac{d^n}{dx^n}$ = nth derivative operator.

Now we can define algebraic operations on operators then selves

If A is an operator, then A^2 does the operation twice

So if $D = \frac{d}{dx}$ then $D^2 =$ take derivative twice $= \frac{d^2}{dx^2}$

and $D^3 = \frac{d^3}{dx^3}, \ldots, D^n = \frac{d^n}{dx^n}$

We can also combine differential operators by addition

$$L = D^2 - 2D + 3$$

this is the operator that multiplies by 3

$$= \frac{d^2}{dx^2} - 2\frac{d}{dx} + 3$$

Then $$L[f(x)] = f''(x) - 2f'(x) + 3f(x)$$

A general constant coefficient linear differential operator looks like

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

$$L \cdot y = \left(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0\right) \cdot y$$

$$= a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

So this is a new notation for the LHS's of the differential equations we want to consider.

Observe the similarity between

Differential operator $\quad L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$

Characteristic polynomial $\quad p(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$

So we can write $L = p(D)$ if we think of D as substituted for r.

Just as we can factor polynomials, we can factor differential operators.

Eg. Factor $\quad D^2 - 2D + 1 = (D-1)^2$

$D^2 + 5D + 1$

$r^2 + 5r + 1 \qquad \dfrac{-5 \pm \sqrt{25-4}}{2} = \dfrac{-5 \pm \sqrt{21}}{2}$

$$D^2 + 5D + 1 = \left[D - \left(\frac{-5+\sqrt{21}}{2}\right)\right]\left[D - \left(\frac{-5-\sqrt{21}}{2}\right)\right]$$

Key fact: For constant coefficient differential operators, the order of the factors does not matter.

$$(D-a)(D-b) = D^2-(a+b)D+ab = (D-b)(D-a)$$

[This isn't true for nonconstant coefficients like $(D-x)$]

Lemma if $L=f(D)g(D)$ for polynomials $f(r)$ and $g(r)$

Then any function y_1 satisfying $f(D)\cdot y=0$ also satisfies $Ly=0$
and any function y_2 satisfying $g(D)\cdot y=0$ also satisfies $Ly=0$

Proof $Ly_1 = f(D)g(D)y_1 = g(D)f(D)y_1$ b/c order doesn't matter here

$$= g(D)\left[f(D)y_1\right] = g(D)[0] = 0$$

And $Ly_2 = f(D)g(D)y_2 = f(D)[g(D)\cdot y_2] = f(D)[0] = 0$.

This is useful b/c it says we can solve the two equations $f(D)y=0$ and $g(D)y=0$ and get solutions to $Ly=0$.

- Case of repeated roots. The "worst case" would be just one root of the characteristic equation

$$p(r) = (r-a)^n = 0 \quad \text{ie} \quad L = p(D) = (D-a)^n$$

Q: What are solutions of $(D-a)^n y = 0$?

We know $y=e^{ax}$ works, since $(D-a)e^{ax} = 0$

Idea: try $f(x)e^{ax}$, and see if it simplifies.

$$(D-a)\left[f(x)e^{ax}\right] = D\left[f(x)e^{ax}\right] - af(x)e^{ax}$$

$$= f'(x)e^{ax} + f(x)ae^{ax} - af(x)e^{ax} = f'(x)e^{ax}$$

$$= \left(D[f(x)]\right)e^{ax}$$

Fact $(D-a)\left[fe^{ax}\right] = (Df)e^{ax}$

Then $(D-a)^2\left[fe^{ax}\right] = (D^2f)\,e^{ax}$

$\vdots$

$(D-a)^n\left[fe^{ax}\right] = (D^n f)\,e^{ax}$

Thus $0=(D-a)^n\left[fe^{ax}\right]$ as long as $D^n f = 0$!

What are the solutions of $D^n f = 0$? Polynomials of degree $n-1$!

$$D^n\left(c_0+c_1x+c_2x^2+\cdots+c_{n-1}x^{n-1}\right) = 0 \ !$$

But $D^n[x^n] = n! \neq 0$

Conclusion: Some solutions of $(D-a)^n y = 0$ are
$y_1 = e^{ax}$, $y_2 = xe^{ax}$, $y_3 = x^2e^{ax}, \ldots, y_n = x^{n-1}e^{ax}$

The general solution is $y(x) = \left(c_0+c_1x+c_2x^2+\cdots+c_{n-1}x^{n-1}\right)e^{ax}$

What if only some roots are repeated?

Eg. $(D-5)(D-2)^2 = (D-5)(D^2-4D+4) = D^3-9D^2+24D-20$

Solutions of $(D-5)y=0$ is $y_1 = e^{5x}$

Solutions of $(D-2)^2 y = 0$ is $y_2 = e^{2x}$ $y_3 = xe^{2x}$

By the lemma, all 3 of these functions solve

$$Ly = (D^3-9D^2+24D-20)y = 0$$

They are linearly independent, so the general solution is

$$\boxed{y(x) = c_1 e^{5x} + c_2 e^{2x} + c_3 x e^{2x}}$$