Nonhomogeneous linear equations, constant coefficient equations

[First we finish showing example of linear independence]

<u>Nonhomogeneous linear diff. eqns.</u>

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

Eg $\quad y'' - 4y = \sin(x)$

We cannot solve using characteristic equation.

Try: $y = e^{rx}$ $\qquad r^2e^{rx} - 4e^{rx} = \sin(x)$? can't work

Later, in section 3.5, we will learn a technique to solve this equation. It turns out that the solution is

$$y_p(x) = -\frac{1}{5}\sin(x)$$

<u>Check</u>: $\quad -\frac{1}{5}(-\sin(x)) - 4\left(-\frac{1}{5}\right)\sin(x) = \dfrac{\sin(x) + 4\sin(x)}{5} = \sin(x)$

<u>Recall</u>: The solutions of $\; y'' - 4y = 0$

are $\quad y_1 = e^{2x}, \; y_2 = e^{-2x}$, with general solution

$$y = c_1e^{2x} + c_2e^{-2x}$$

So the solution of the nonhomogeneous equation looks completely different from that of the homogeneous one.

The principle of superposition does not hold for nonhomogeous equations.

so for example $2\cdot\left(-\frac{1}{5}\sin(x)\right)$ is not a solution

But a related principle does apply

If we add a solution of the homogeneous equation to a solution of the nonhomogeneous equation, we get a solution of the non homogeneous equation

Let y_p be a solution of the nonhomogeneous equation

$$y''+p(x)y'+q(x)y=f(x)$$

"particular solution"

So let y_c be a solution of the homogeneous equation

$$y''+p(x)y'+q(x)y=0$$

Then y_c+y_p satisfies $y''+p(x)y'+q(x)y=f(x)$

* This also works for higher order equations.

- Possible analogy:

  Even and odd numbers

  homog solutions (Even) ↕ non homog. solutions (odd)

  Even + Even = Even

  Odd + Even = Odd

  Odd + Odd ≠ Odd

  Odd + Odd = even

  This is where the analogy breaks down

  So don't take it too seriously

Proof $(y_c+y_p)'' + p(x)(y_c+y_p)' + q(x)(y_c+y_p)$

$$= y_c'' + y_p'' + p(x)\, y_c' + p(x) y_p' + q(x)\, y_c + q(x)\, y_p$$

$$= \left[y_c'' + p(x) y_c' + q(x) y_c\right] + \left[y_p'' + p(x) y_p' + q(x)\, y_p\right]$$

$$= \quad 0 \quad + \quad f(x)$$

$$= f(x)$$

QED.

General solution Consider nth order linear nonhomogeneous DE

$$y^{(n)} + p_1(x)\, y^{(n-1)} + \cdots + p_n(x) y = f(x)$$

If y_p is any particular solution of this equation,

and $y_1, y_2, \ldots, y_n$ are n linearly independent solutions

of the homogeneous equation

$$y^{(n)} + p_1(x)\, y^{(n-1)} + \cdots + p_n(x) y = 0$$

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

is the general solution of the nonhomogeneous equation.

Example: $y'' - 4y = \sin(x)$

(a) Find general solution.

(b) Solve initial value problem $y(0) = 0$, $y'(0) = 0$

a Particular solution is known $y_p(x) = \frac{-1}{5}\sin(x)$

The general solution of the homogeneous equation $y'' - 4y = 0$ is known:

$$y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$$

So the general solution of $y'' - 4y = \sin(x)$ is

$$y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{5}\sin(x)$$

Solve initial value problem

$$0 = y(0) = c_1 e^0 + c_2 e^0 - \frac{1}{5}\sin(0) = c_1 + c_2$$

$$y'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x} - \frac{1}{5}\cos(x)$$

$$0 = y'(0) = 2c_1 e^0 - 2c_2 e^0 - \frac{1}{5}\cos(0) = 2c_1 - 2c_2 - \frac{1}{5}$$

$c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$2c_1 - 2c_2 = \frac{1}{5} \longrightarrow 4c_1 = \frac{1}{5}$ $\quad c_1 = \frac{1}{20}$ $\quad c_2 = \frac{-1}{20}$

$$\boxed{y(x) = \frac{1}{20}e^{2x} - \frac{1}{20}e^{-2x} - \frac{1}{5}\sin(x)}$$