Second order linear equations continued.

Without further delay, let's see how to solve a 2nd order linear homogeneous equation.

We can do this when the equation has constant coefficients

Consider $ay'' + by' + cy = 0$

What are some solutions?

Recall 1st order case $ay' + by = 0$

$$\frac{y'}{y} = -\frac{b}{a} \longrightarrow \int \frac{dy}{y} = \int \left(-\frac{b}{a}\right) dx$$

$$\ln|y| = \left(-\frac{b}{a}\right) x + C \longrightarrow y = D e^{\left(-\frac{b}{a}\right)x}$$

So, let's try $y(x) = e^{rx}$, where r is a constant to be determined.

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}$$

So $ay'' + by' + cy = 0$ becomes

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

Since e^{rx} is never zero, we can divide by it:

$$ar^2 + br + c = 0.$$

This is the <u>characteristic equation</u>. It is the condition that r must satisfy in order for $y = e^{rx}$ to solve the original DE.

<u>Example</u>: $y'' - 4y' + 3y = 0$

Try e^{rx}

$$r^2 e^{rx} - 4re^{rx} + 3e^{rx} = 0$$
$$r^2 - 4r + 3 = 0$$
$$(r-3)(r-1) = 0$$

so $r = 3$ or 1.

Thus $y_1 = e^{3x}$ and $y_2 = e^{1x} = e^x$ are solutions!

Now by the principle of superposition for linear homogeneous equations, we know that

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 e^x$$

is also a solution, for any constants c_1 and c_2.

In general, if r_1 and r_2 are solutions of the characteristic equation $ar^2 + br + c = 0$, then

And the solutions are real and distinct,

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a solution of

$$ay'' + by' + cy = 0$$

The question is, how do we know when we've found "enough" solutions?

We start with a theoretical fact

<u>Theorem</u> (Existence and uniqueness for 2nd order linear equations) Suppose $p(x)$, $q(x)$, and $f(x)$ are continuous on an interval I and let a be a point of I.

THEN the initial value problem

$$\left\{ \begin{array}{l} y'' + p(x)y' + q(x)y = f(x) \\ y(a) = b_0 \\ y'(a) = b_1 \end{array} \right\}$$

has a unique solution defined on I.

Rephrasing, we should get a unique solution for any <u>pair</u> of initial value and initial derivative.

We have found "enough" solutions when we have enough constants to solve any such initial value problem.

Suppose we find two solutions $y_1(x)$ and $y_2(x)$ to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$.

Need to be able to solve for

$$\begin{array}{l} y(a) = b_0 \\ y'(a) = b_1 \end{array}$$

or

$$\left\{ \begin{array}{l} c_1 y_1(a) + c_2 y_2(a) = b_0 \\ c_1 y_1'(a) + c_2 y_2'(a) = b_1 \end{array} \right\}$$

This is a 2×2 linear system for c_1 and c_2

The coefficient matrix is $\begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix}$

The system is always solvable if it is nonsingular, which means the determinant.

$$\begin{vmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{vmatrix} = y_1(a)y_2'(a) - y_2(a)y_1'(a) \text{ } \underline{\text{is not zero}}.$$

This determinant is called the Wronskian of y_1 and y_2 at a.

It turns out that this condition is equivalent to saying y_1 and y_2 are <u>linearly independent</u>.

<u>Def</u> Two functions y_1 and y_2 are <u>linearly independent</u> if they are not proportional:

$$y_1 \neq C y_2$$
$$y_2 \neq C y_1$$

Otherwise they are <u>linearly dependent</u>.

<u>Examples</u> $\sin(x), \cos(x)$; e^x, e^{-2x} ; e^x, xe^x , ...

<u>Non-examples</u>: $e^x, 2e^x$; $0, \sin(x)$, ...

General solution of second order linear homogeneous:

If y_1 and y_2 are linearly independent solutions of

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Then the general solution of this equation is

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

That is to say, <u>every</u> solution is of this form, and you can solve <u>any</u> initial value problem by picking particular values for the constants C_1 and C_2.