Differential equations

Q1: What are they?
Q2: Why would you care?
Q3: What are we doing in this course?

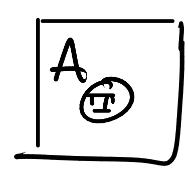
A1: A differential equation is an equation relating the derivatives of some quantities to other quantities.

A2: Since a derivative represents a rate of change, any quantitative problem involving rates of change necessarily involves differential equations of some kind.

A3: Because the concept of a differential equation is so broad, and touches so many disparate phenomena, we cannot solve all DEs. Instead we focus on specific cases that are tractable and that have interesting applications.

<u>Example</u>: Newton's law of cooling

An object is immersed in a bath held at a fixed temperature A (ambient temperature)



Denote by T the temperature of the object. This is actually a function of time (t), since the object will cool or heat up to match the ambient temp A (tends toward thermodynamic equilibrium)

Newton says

$$\frac{dT}{dt} = -k(T-A)$$

k = constant measuring rate at which heat is conducted to/from the object. Depends on physical composition of the object. k is always positive

This is a differential equation because it involves the derivative $\frac{dT}{dt}$.

We want to know more explicitly what T is as a function of T. In fact

$$T(t) = Ce^{-kt} + A$$

<u>Solves</u> the Differential equation, where C is <u>any number</u>.

It's easy to check. Let $T(t) = Ce^{-kt} + A$

$$LHS = \frac{dT}{dt} = \frac{d}{dt}\left(Ce^{-kt}\right) + \frac{d}{dt}(A) = C\frac{d}{dt}\left(e^{-kt}\right) + 0$$

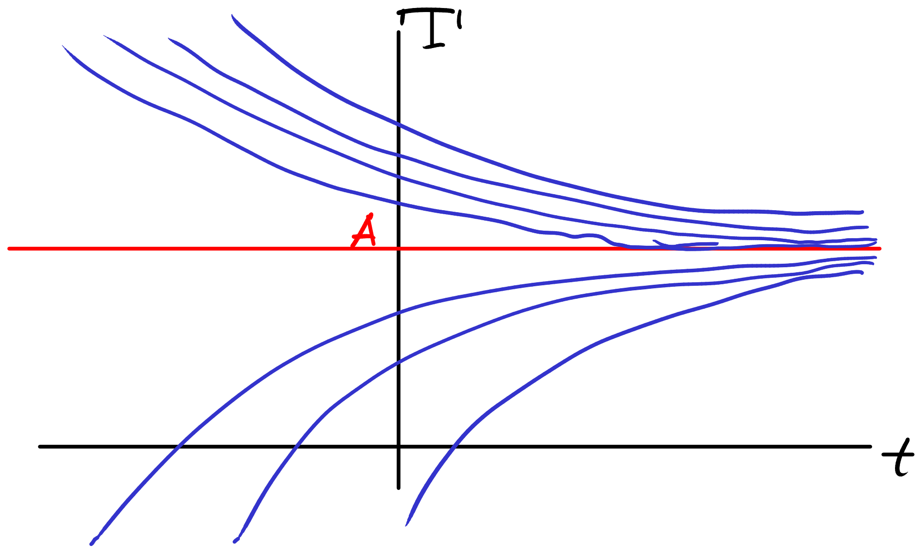
$$= C(-k)e^{-kt}$$

equal!

$$RHS = -k(T-A) = -k\left(Ce^{-kt} + A - A\right) = -kCe^{-kt}$$

Observe: the DE has infinitely many solutions, because there are infinitely many possibilities for the constant C. This is a typical characteristic.

Plots of possible solutions



Makes sense that $T(t)$ is not unique, because we never decided whether the object starts out above or below the Ambient temp. A.

In other words, we need to specify the initial temperature $T_0 = T(0)$.

T_0 is related to C:

$$T_0 = T(0) = Ce^{-k\cdot 0} + A = C + A$$

So $C = T_0 - A$.

So $T(t) = (T_0 - A)e^{-kt} + A$

We solved the initial value problem $\frac{dT}{dt} = -k(T - A)$, $T(0) = T_0$.

To be more specific $\quad t$ = time in seconds

T = temp in Kelvin

$k = 5 \ s^{-1}$

$A = 300 \ K$

$T_0 = 2000 \ K$

$$T(t) = (T_0 - A)e^{-kt} + A$$
$$= 1700\, e^{-5t} + 300$$

What differential equation does it solve?

$y = e^t \qquad \frac{dy}{dt} = e^t = y \qquad \boxed{\frac{dy}{dt} = y}$

$y = \sin t \qquad \frac{dy}{dt} = \cos t \qquad \boxed{\frac{d^2y}{dt^2} = -y}$

$\frac{d^2y}{dt^2} = -\sin t = -y$

$y = \frac{1}{1-x} \qquad \frac{dy}{dx} = \frac{-1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2} = y^2$

$\boxed{\frac{dy}{dx} = y^2}$

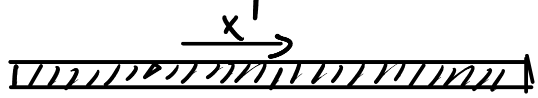
$y = \frac{1}{(1-x)^2} \qquad \frac{dy}{dx} = \frac{2}{(1-x)^3} = 2\,y^{3/2} \qquad \boxed{\frac{dy}{dx} = 2\,y^{3/2}}$

What about the equation $(y')^2 + (y)^2 = -1$?
No (real) solution!

So existence of solutions is sometimes in doubt.

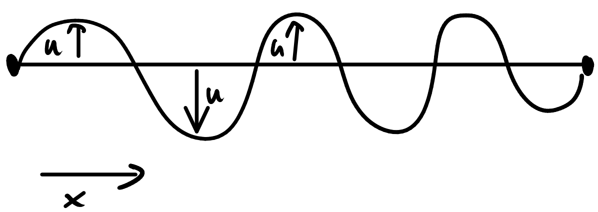
Many problems involve functions of several variables
Then we are dealing with <u>partial differential equations</u>

<u>Heat conduction</u>: $T(x,t)$ temperature in rod



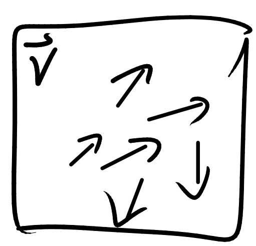
Heat Equation : $$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

<u>Wave propagation</u>: $u(x,t)$ = displacement of vibrating string



Wave equation $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

<u>Fluid flow</u> $\vec{v}$ = velocity vector field
p = pressure
k = viscosity $\vec{f}$ = external force



(1) $$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = k \nabla^2 \vec{v} - \nabla p + \vec{f}$$

(2) $\nabla \cdot \vec{v} = 0$

(3) $\vec{v}(x,0) = \vec{v}_0(x)$ initial condition

Navier-Stokes
\$1000000 prize
see claymath.org