MATH 285 E1/F1 GRADED HOMEWORK SET 7
DUE MONDAY DECEMBER 8 IN LECTURE

This time, the homework has **just one part**. Please staple your homework together, and put your **name and section** on it. *Thank you!*

(1) (15 points) Find the solution of following wave problem for vibrations in a string of length 10 (corresponding to a struck string).

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 16\frac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ u(10,t) = 0 \\ u(x,0) = 0 \\ \frac{\partial u}{\partial t}(x,0) = x(10-x) \end{cases}$$

Although this problem was essentially done in lecture for general initial conditions, we will recall the solution. The separable solutions $u(x,t) = X(x)T(t)$ of the wave equation and the boundary conditions satisfy

$$\frac{d^2X}{dx^2} + \lambda X = 0, \quad X(0) = 0, \quad X(10) = 0$$

$$\frac{d^2T}{dt^2} + c^2\lambda T = 0$$

We recognize the conditions on X as an eigenvalue problem, with eigenvalues $\lambda_n = (n\pi/10)^2$ and eigenfunctions $X_n(x) = \sin(n\pi x/10)$ where $n = 1, 2, 3, \ldots$ ranges over the positive integers. The corresponding T_n solves

$$\frac{d^2T_n}{dt^2} + c^2(n\pi/10)^2 T_n = 0$$

whence

$$T_n(t) = A_n\cos(cn\pi t/10) + B_n\sin(cn\pi t/10)$$

The separable solutions are

$$u_n(x,t) = [A_n\cos(cn\pi t/10) + B_n\sin(cn\pi t/10)]\sin(n\pi x/10)$$

and the general solution is

$$u(x,t) = \sum_{n=1}^{\infty}[A_n\cos(cn\pi t/10) + B_n\sin(cn\pi t/10)]\sin(n\pi x/10)$$

Now to apply the initial conditions. Plugging $t = 0$ into the general solution:

$$u(x, 0) = \sum_{n=1} A_n \sin(n\pi x/10)$$

The initial condition says that this ought to be zero, so we conclude that all A_n are zero. Thus

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin(cn\pi t/10) \sin(n\pi x/10)$$

where have yet to determine B_n. Differentiate with respect to t:

$$\frac{\partial u}{\partial t}(x, t) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{10} \cos(cn\pi t/10) \sin(n\pi x/10)$$

Plugging in $t = 0$ makes the cosine factors become 1, and we get something that looks like a sine series:

$$\frac{\partial u}{\partial t}(x, 0) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{10} \sin(n\pi x/10)$$

The initial condition says that this ought to equal $x(10 - x)$. So we need to find the sine series of this function on the interval $0 < x < 10$. Thus we must compute

$$b_n = \frac{2}{10} \int_0^{10} x(10 - x) \sin(n\pi x/10)\, dx$$

This can be done directly by repeated integration by parts. The result is

$$b_n = \frac{400}{(n\pi)^3}(1 - \cos n\pi) = \begin{cases} \frac{800}{(n\pi)^3} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

We have the relation

$$B_n \frac{cn\pi}{10} = b_n$$

so

$$B_n = \begin{cases} \frac{8000}{c(n\pi)^4} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

Noting that actually $c = 4$ (which could have been put in earlier if desired):

$$u(x, t) = \sum_{n \text{ odd}} \frac{2000}{(n\pi)^4} \sin(4n\pi t/10) \sin(n\pi x/10)$$

(2) (5 points) The function

$$u(x,t) = \sin(20t)\cos(5x)$$

solves a wave equation. Write $u(x,t)$ as a sum of a right-moving wave and a left-moving wave, and determine the speed of these waves. *Hint:* Scour the article "List of trigonometric identities" on Wikipedia for a relevant identity.

The relevant identity is what is called the product-to-sum formula

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

Thus

$$u(x,t) = \sin(20t)\cos(5x) = \frac{1}{2}\sin(20t+5x) + \frac{1}{2}\sin(20t-5x)$$

We would like to see expressions like $x - ct$ and $x + ct$ inside the sines, so we factor out the 5, and use the fact that $\sin(-z) = -\sin z$:

$$u(x,t) = \frac{1}{2}\sin(5(x+4t)) - \frac{1}{2}\sin(5(x-4t))$$

Thus if we define $F(z) = \frac{1}{2}\sin 5z$, we find

$$u(x,t) = F(x+4t) - F(x-4t)$$

The term $F(x+4t) = \frac{1}{2}\sin(5(x+4t))$ is a left-moving wave with speed $c = 4$, and the term $-F(x-4t) = -\frac{1}{2}\sin(5(x-4t))$ is a right-moving wave with speed 4.

(3) (15 points) Find the solution of the Laplace equation problem on the square $0 \le x \le 1$, $0 \le y \le 1$.

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ u(x,0) = 1 \\ u(x,1) = 0 \\ u(0,y) = 0 \\ u(1,y) = 0 \end{cases}$$

Taking into account Laplace's equation and the homogeneous boundary conditions, we find that the separable solutions $u(x,y) = X(x)Y(y)$ satisfy

$$\frac{d^2X}{dx^2} + \lambda X = 0, \quad X(0) = 0, \quad X(1) = 0,$$

$$\frac{d^2Y}{dy^2} - \lambda Y = 0, \quad Y(1) = 0.$$

The set of conditions on X is an eigenvalue problem, with solutions $\lambda_n = (n\pi)^2$, $X_n(x) = \sin n\pi x$, where $n = 1, 2, 3, \ldots$. The corresponding function $Y_n(y)$ must satisfy

$$\frac{d^2Y_n}{dy^2} - (n\pi)^2 Y_n = 0$$

So

$$Y_n(y) = Ae^{n\pi y} + Be^{-n\pi y}$$

The condition $Y_n(1) = 0$ means

$$0 = Ae^{n\pi} + Be^{-n\pi}$$

So we can write $B = -Ae^{2n\pi}$, and thus

$$Y_n(y) = Ae^{n\pi y} - Ae^{2n\pi}e^{-n\pi y} = A(e^{n\pi y} - e^{n\pi(2-y)})$$

This is actually good enough to work with, but it can also be written in terms of sinh. Let's factor out a factor of $e^{n\pi}$:

$$Y_n(y) = Ae^{n\pi}(e^{n\pi(y-1)} - e^{n\pi(1-y)}) = -2Ae^{n\pi}\sinh[n\pi(1-y)]$$

Since $-2Ae^{n\pi}$ is just a constant, we can take

$$u_n(x, y) = \sinh[n\pi(1-y)]\sin n\pi x$$

as our basic separable solution. The general solution to Laplace's equation and the three homogeneous boundary conditions is therefore

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh[n\pi(1-y)]\sin n\pi x$$

Last is to satisfy the condition $u(x, 0) = 1$. Plugging in $y = 0$, we get

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sinh n\pi \sin n\pi x$$

which has the form of a sine series with $L = 1$. Thus we need to take the sine series of 1 on the interval $0 < x < 1$, the coefficients are given by

$$b_n = \frac{2}{1}\int_0^1 1 \sin n\pi x\, dx = \frac{2}{n\pi}(1 - \cos n\pi) = \begin{cases} \frac{4}{n\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

These coefficients are related to c_n by

$$c_n \sinh n\pi = b_n$$

Thus

$$c_n = \begin{cases} \frac{4}{n\pi \sinh n\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

We conclude

$$u(x, y) = \sum_{n \text{ odd}} \frac{4}{n\pi} \frac{\sinh[n\pi(1-y)]}{\sinh n\pi} \sin n\pi x$$