MATH 285 E1/F1 GRADED HOMEWORK SET 6
DUE FRIDAY NOVEMBER 21 IN LECTURE

This time, the homework has **just one part**. Please staple your homework together, and put your **name and section** on it. *Thank you!*

(1) (15 points) Consider the eigenvalue problem

$$\begin{cases} y'' + 2y' + \lambda y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$$

Find the eigenvalues and eigenfunctions for this problem. That is, find the values of λ for which the problem has a nontrivial solution, and find those nontrivial solutions. *Hint:* The smallest eigenvalue is $\pi^2 + 1$, with associated eigenfunction $e^{-x} \sin \pi x$. (In your answer you should verify this.)

Considering first the differential equation, the corresponding characteristic equation is

$$r^2 + 2r + \lambda = 0$$

with solutions

$$r = \frac{-2 \pm \sqrt{4 - 4\lambda}}{2} = -1 \pm \sqrt{1 - \lambda}$$

Thus the nature of the solutions depends on whether $1 - \lambda > 0$, $1 - \lambda = 0$, or $1 - \lambda < 0$.

First consider the case $1 - \lambda > 0$. Then the function y has the form

$$y = Ae^{(-1+\sqrt{1-\lambda})x} + Be^{(-1-\sqrt{1-\lambda})x}$$

We ask what restriction the endpoint conditions place on A and B:

$$y(0) = 0 \implies 0 = A + B$$

$$y(1) = 0 \implies 0 = Ae^{-1+\sqrt{1-\lambda}} + Be^{-1-\sqrt{1-\lambda}}$$

If we multiply the latter equation by $e^{1+\sqrt{1-\lambda}}$, we obtain

$$0 = Ae^{2\sqrt{1-\lambda}} + B$$

In conjunction with $0 = A + B$, we obtain $A = Ae^{2\sqrt{1-\lambda}}$. This can only happen if $A = 0$. But then $B = 0$ as well. Thus the only solution is $y = 0$, the trivial/uninteresting solution. We conclude that λ is not an eigenvalue.

Now we consider the case $1-\lambda = 0$, which is to say $\lambda = 1$. The characteristic roots $r = -1 \pm \sqrt{1-\lambda}$ are both equal to -1, so we have a repeated root. Then the function y has the form

$$y = Ae^{-x} + Bxe^{-x}$$

Considering the endpoint conditions:

$$y(0) = 0 \implies 0 = A$$

$$y(1) = 0 \implies 0 = Ae^{-1} + Be^{-1}$$

Thus we find directly $A = 0$, and hence $0 = Be^{-1}$, so $B = 0$ as well. Thus the only solution is $y = 0$, and we conclude that λ is not an eigenvalue.

Lastly, we consider the case $1-\lambda < 0$. Since now $\lambda - 1$ is a positive number, the characteristic roots may be written:

$$r = -1 \pm \sqrt{1-\lambda} = -1 \pm i\sqrt{\lambda - 1}$$

Then the function y has the form

$$y = Ae^{-x}\cos\sqrt{\lambda-1}x + Be^{-x}\sin\sqrt{\lambda-1}x$$

Considering the endpoint conditions:

$$y(0) = 0 \implies 0 = A$$

$$y(1) = 0 \implies 0 = Ae^{-x}\cos\sqrt{\lambda-1} + Be^{-1}\sin\sqrt{\lambda-1}$$

The first equation gives directly that $A = 0$, so the second one simplifies to

$$0 = Be^{-1}\sin\sqrt{\lambda-1}$$

The number e^{-1} is not zero, so this equation implies that either $B = 0$ or $\sin\sqrt{\lambda-1} = 0$.

Now we have two subcases, either $\sin\sqrt{\lambda-1} = 0$ or not. If not, then $B = 0$, so the entire solution $y = 0$, and we conclude that λ is not an eigenvalue. But, if the number λ does satisfy the condition

$$\sin\sqrt{\lambda-1} = 0,$$

then B can take any value, and we have an nontrivial/interesting solution, namely $y = Be^{-x}\sin\sqrt{\lambda-1}x$, so λ is an eigenvalue.

Thus the condition for λ to be an eigenvalue is

$$\sin\sqrt{\lambda-1} = 0$$

This is equivalent to

$$\sqrt{\lambda-1} = n\pi, \quad n = 1, 2, 3, \ldots$$

$$\lambda = (n\pi)^2 + 1, \quad n = 1, 2, 3, \ldots$$

The eigenfunction associated to the eigenvalue $(n\pi)^2 + 1$ is

$$y = e^{-x}\sin n\pi x$$

(2) (10 points) Find the solution of the heat problem on the interval $0 \le x \le 5$:

$$\begin{cases} \frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \\ \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(5,t) = 0 \\ u(x,0) = \cos^2 10\pi x \end{cases}$$

This is the heat equation problem with insulated ends, which was discussed in lecture (see lecture 33). It corresponds to the case $k = 2$ and $L = 5$. We quote from that lecture: the eigenfunctions are the constant function 1 and the cosine functions $\cos \frac{n\pi x}{5}$. The separable solutions are $u_n(x,t) = e^{-2(n\pi/5)^2 t} \cos \frac{n\pi x}{5}$. The general solutions is

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-2(n\pi/5)^2 t} \cos \frac{n\pi x}{5}$$

with initial value

$$u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{5}$$

We need this to match $f(x) = \cos^2 10\pi x$, so we need to find the Fourier cosine series (with period 10, $L = 5$) of $f(x)$. That can be done by using the double angle formula $\cos 2\theta = 2\cos^2\theta - 1$, or $\cos^2\theta = (1 + \cos 2\theta)/2$

$$f(x) = \cos^2 10\pi x = (1 + \cos 20\pi x)/2 = \frac{1}{2} + \frac{1}{2}\cos\frac{100\pi x}{5}$$

Thus, in order to match the initial data, we must take $c_0 = 1/2$, $c_{100} = 1/2$, and all other coefficients zero

$$c_0 = 1/2, c_1 = 0, c_2 = 0, \ldots, c_{99} = 0, c_{100} = 1/2, c_{101} = 0, \ldots$$

Putting it all together, the solution is

$$u(x,t) = \frac{1}{2} + \frac{1}{2} e^{-2(100\pi/5)^2 t} \cos\frac{100\pi x}{5}$$

or, simplified:

$$u(x,t) = \frac{1}{2} + \frac{1}{2} e^{-800\pi^2 t} \cos 20\pi x$$

(3) (15 points) Find the solution of the heat problem on the interval $0 \le x \le 1$:

$$\begin{cases} \frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ u(1,t) = 1 \\ u(x,0) = x^2 \end{cases}$$

You should use the following strategy:

(a) Find a steady-state solution u_0 of all parts of the problem except the initial condition. That is, find a function $u_0(x,t)$ that is constant in time ($\frac{\partial u}{\partial t} = 0$), that satisfies the heat equation, and that satisfies the conditions $u_0(0,t) = 0$ and $u_0(1,t) = 1$.
Assuming $\frac{\partial u_0}{\partial t} = 0$ and the heat equation, we obtain $\frac{\partial^2 u_0}{\partial x^2} = 0$, so $u_0(x,t)$ must be independent of t and linear in x

$$u_0(x,t) = Ax + B$$

In order to satisfy the boundary conditions, we must have $0 = u_0(0,t) = B$, and $1 = u_0(1,t) = A+B$, hence $B = 0$ and $A = 1$. Thus the steady state solution is

$$u_0(x,t) = x$$

(b) Posit $w = u - u_0$, and show that w must solve a slighty different heat problem:

$$\begin{cases} \frac{\partial w}{\partial t} = 5\frac{\partial^2 w}{\partial x^2} \\ w(0,t) = 0 \\ w(1,t) = 0 \\ w(x,0) = x^2 - u_0(x,0) \end{cases}$$

(w is called the transient term).
We are assuming that $u(x,t)$ solves the problem, and we want to determine the relevant condtions on $w(x,t)$. First of all, w satisfies the heat equation:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}(u - u_0) = \frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = 5\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u_0}{\partial x^2} = 5\frac{\partial^2}{\partial x^2}(u - u_0) = 5\frac{\partial^2 w}{\partial x^2}$$

Next, w satisfies $w(0,t) = 0$:

$$w(0,t) = u(0,t) - u_0(0,t) = 0 - 0 = 0$$

Next, w satisfies $w(1,t) = 0$:

$$w(1,t) = u(1,t) - u_0(1,t) = 1 - 1 = 0$$

Lastly, w satisfies $w(x,0) = x^2 - u_0(x,0)$:

$$w(x,0) = u(x,0) - u_0(x,0) = x^2 - u_0(x,0)$$

(c) Using the methods described in the lectures, find the solution $w(x,t)$, and hence $u(x,t)$.
Using the steady state solution $u_0(x,0) = x$, we need to solve the heat problem

$$\begin{cases} \frac{\partial w}{\partial t} = 5\frac{\partial^2 w}{\partial x^2} \\ w(0,t) = 0 \\ w(1,t) = 0 \\ w(x,0) = x^2 - x \end{cases}$$

We now quote from lecture 32 on this problem, with $k = 5$ and $L = 1$: the general solution is a the series

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-5(n\pi)^2 t} \sin n\pi x$$

and

$$w(x,0) = \sum_{n=1}^{\infty} c_n sin n\pi x$$

So we need to compute the Fourier sine coefficients of $x^2 - x$. These are

$$b_n = 2\int_0^1 (x^2 - x)\sin n\pi x\, dx$$

We use integration by parts twice

$$b_n = 2\left[(x^2-x)(-1/n\pi)\cos n\pi x\right]_0^1 - 2\int_0^1 (2x-1)(-1/n\pi)\cos n\pi x\, dx$$

$$= 2[0-0] + (2/n\pi)\int_0^1 (2x-1)\cos n\pi x\, dx$$

$$= (2/n\pi)\left\{[(2x-1)(1/n\pi)\sin n\pi x]_0^1 - \int_0^1 2(1/n\pi)\sin n\pi x\, dx\right\}$$

$$= (2/n\pi)\left\{[0-0] + \left[2(1/n\pi)^2\cos n\pi x\right]_0^1\right\}$$

$$= (4/(n\pi)^3)(\cos n\pi - 1) = (4/(n\pi)^3)((-1)^n - 1)$$

We can see that this expression is 0 if n is even, and equal to $-8/(n\pi)^3$ if n is odd. Thus the Fourier sine series of $x^2 - x$ on the interval $0 < x < 1$ is

$$x^2 - x = \sum_{n \text{ odd}} \frac{-8}{(n\pi)^3}\sin n\pi x$$

To get the solution of the heat equation we set $c_n = b_n$ in the general solution

$$w(x,t) = \sum_{n \text{ odd}} \frac{-8}{(n\pi)^3} e^{-5(n\pi)^2 t}\sin n\pi x$$

This is the solution for the w-problem. To get the solution the original problem for u, we add back $u_0 = x$:

$$u(x,t) = u_0(x,t) + w(x,t) = x + \sum_{n \text{ odd}} \frac{-8}{(n\pi)^3} e^{-5(n\pi)^2 t}\sin n\pi x$$