MATH 285 E1/F1 GRADED HOMEWORK SET 5
DUE WEDNESDAY NOVEMBER 5 IN LECTURE

This time, the homework has **just one part**. Please staple your homework together, and put your **name and section** on it. *Thank you!*

(1) (15 points, from [1, p. 190]) Expand x^3 and x in Fourier sine series valid when $-\pi < x < \pi$; and hence find the value of the sum of the series

$$\sin x - \frac{1}{2^3}\sin 2x + \frac{1}{3^3}\sin 3x - \frac{1}{4^3}\sin 4x + \cdots$$

for all values of x.[1]

Since both x and x^3 are odd functions, the constant term and all of the cosine terms of their Fourier series vanish, so the Fourier series contains only sine terms (as the problem statement says). Again using the odd symmetry, the sine coefficients are computed by the integrals

$$\frac{2}{\pi}\int_0^\pi x \sin nx\,dx, \qquad \frac{2}{\pi}\int_0^\pi x^3 \sin nx\,dx$$

Let's consider the function x first (this was actually done in lecture). We use integration by parts

$$\begin{aligned}\int x\sin nx\,dx &= -(1/n)x\cos nx - \int -(1/n)\cos nx\,dx \\ &= -(1/n)x\cos nx + (1/n^2)\sin nx + C\end{aligned}$$

Plugging in the limits of integration 0 and π gives

$$\begin{aligned}b_n &= \frac{2}{\pi}\int_0^\pi x\sin nx\,dx \\ &= \frac{2}{\pi}[-(1/n)\pi\cos n\pi + (1/n^2)\sin n\pi + (1/n)0\cos 0 - (1/n^2)\sin 0] \\ &= \frac{2}{\pi}[-(1/n)\pi\cos n\pi] = \frac{2}{n}(-1)^{n+1}\end{aligned}$$

Thus, on the interval $-\pi < x < \pi$, we have the equality

$$x = \sum_{n=1}^\infty b_n \sin nx = 2\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sin nx$$

[1]According to Whittaker and Watson [1], this problem was on an exam at Jesus College, Cambridge, in 1902.

Now let's consider the function x^3. We reuse the symbol b_n to denote the sine coefficients of x^3. We need to integrate $x^3 \sin nx$, and for this we use integration by parts recursively.

$$\begin{aligned}\int x^3 \sin nx\, dx &= x^3(-1/n)\cos nx - \int 3x^2(-1/n)\cos nx\, dx \\ &= -(1/n)x^3 \cos nx + (3/n)\int x^2 \cos nx\, dx\end{aligned}$$

$$\begin{aligned}\int x^2 \cos nx\, dx &= x^2(1/n)\sin nx - \int 2x(1/n)\sin nx\, dx \\ &= (1/n)x^2 \sin nx - (2/n)\int x \sin nx\, dx\end{aligned}$$

$$\int x \sin nx\, dx = -(1/n)x\cos nx + (1/n^2)\sin nx + C$$

Putting it together,

$$\begin{aligned}&\int x^3 \sin nx\, dx \\ &= -(1/n)x^3\cos nx + (3/n)\{(1/n)x^2 \sin nx - (2/n)[-(1/n)x\cos nx + (1/n^2)\sin nx]\} + C\end{aligned}$$

Rather than multiplying this out, let's just see what happens when we plug in 0 and π. If we plug in 0, the the terms $x^3 \cos nx$, $x^2 \sin nx$, $x \cos nx$, and $\sin nx$ all become zero, so the whole expression is zero. If we plug in π, the terms $x^2 \sin nx$ and $\sin nx$ become zero, so we are left with

$$\begin{aligned}&-(1/n)\pi^3 \cos n\pi + (3/n)\{-(2/n)[-(1/n)\pi \cos n\pi]\} \\ &= -(1/n)\pi^3(-1)^n + (6/n^3)\pi(-1)^n\end{aligned}$$

Thus

$$\begin{aligned}b_n &= \frac{2}{\pi}\int_0^\pi x^3 \sin nx\, dx = \frac{2}{\pi}[(-1)^{n+1}(1/n)\pi^3 + (-1)^n(6/n^3)\pi] \\ &= (-1)^{n+1}(2/n)\pi^2 + (-1)^n(12/n^3)\end{aligned}$$

Thus, for $-\pi < x < \pi$, we have

$$x^3 = \sum_{n=1}^{\infty}[(-1)^{n+1}(2/n)\pi^2 + (-1)^n(12/n^3)]\sin nx$$

This completes the first part of the problem.

The second part is to find the sum of the given series. Call that function $f(x)$. We notice that this series may be written using $\sum$ notation as

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}(1/n^3)\sin nx$$

The sine coefficients of x are $P_n = (-1)^{n+1}(2/n)$, those of x^3 are $Q_n = (-1)^{n+1}(2/n)\pi^2 + (-1)^n(12/n^3)$, and those of $f(x)$ are $R_n = (-1)^{n+1}(1/n^3)$. There is a relationship between these coefficients

$$Q_n = \pi^2 P_n - 12R_n$$

and thus a relationship between the corresponding functions

$$x^3 = \pi^2 x - 12f(x)$$

Thus, for $\pi < x < \pi$

$$f(x) = (1/12)(\pi^2 x - x^3)$$

The function $f(x)$ repeats periodically with period 2π, and it is in fact continuous.

(2) (5 points) Find the Fourier cosine series of the function $f(t) = 1 - t$ defined on the interval $0 < t < 1$.

Recall that the cosine series is defined by taking the Fourier series of the even extension of period $2L = 2$. The Fourier cosine coefficients are

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt = 2\int_0^1 (1-t)\,dt = 2[t - t^2/2]_0^1 = 2(1-1/2) = 1$$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt = 2\int_0^1 (1-t)\cos n\pi t\,dt$$

The integral is done by parts

$$\begin{aligned}\int (1-t)\cos n\pi t\,dt &= (1-t)\frac{1}{n\pi}\sin n\pi t - \int (-1)\frac{1}{n\pi}\sin n\pi t\,dt \\ &= \frac{1}{n\pi}(1-t)\sin n\pi t - \frac{1}{(n\pi)^2}\cos n\pi t + C\end{aligned}$$

$$\begin{aligned}a_n &= 2\left[\frac{1}{n\pi}(1-t)\sin n\pi t - \frac{1}{(n\pi)^2}\cos n\pi t\right]_0^1 \\ &= 2\left[\frac{1}{n\pi}(1-1)\sin n\pi - \frac{1}{(n\pi)^2}\cos n\pi - \frac{1}{n\pi}(1-0)\sin 0 + \frac{1}{(n\pi)^2}\cos 0\right] \\ &= \frac{2}{(n\pi)^2}[\cos 0 - \cos n\pi] = \frac{2}{(n\pi)^2}[1 - (-1)^n]\end{aligned}$$

Thus $a_n = 0$ if n is even, and $a_n = 4/(n\pi)^2$ if n is odd.

The cosine series is then

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}\cos n\pi t$$

(3) (10 points) Let $f(t)$ be the periodic function of period 2 defined on the interval $0 < t < 2$ by the formula $f(t) = t^2$. The Fourier series of this function is

$$\frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi t - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi t$$

(a) (5 points) Describe precisely the value of the sum of the Fourier series at every value of t, including at the points of discontinuity of the original function $f(t)$.

The function $f(t)$ is continuous except at the points $t = 2k$, where k is an integer. At each such point, the limit from the left is $2^2 = 4$, while the limit from the right is $0^2 = 0$. So the Fourier series converges to $(4+0)/2 = 2$ at these points.

At any point t not of the form $2k$, where k is an integer, the Fourier series converges to s^2, where s is the number in the interval $0 < s < 2$ obtained by adding a multiple of 2 to t. If $[x]$ denotes the greatest integer less than or equal to x, then $s = t - 2[t/2]$. In other words,

$$\frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi t - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi t = \begin{cases} 2 & t = 2k \\ (t-2[t/2])^2 & t \neq 2k \end{cases}$$

(b) (5 points) Suppose we differentiate the Fourier series term-by-term. Show that the resulting series does not converge (to anything, in particular not to $f'(t)$). *Hint:* try to plug $t = 1/2$ into the differentiated series.

Taking the derivative term-by-term yields

$$\begin{aligned} &\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-n\pi)\sin n\pi t - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}n\pi\cos n\pi t \\ &= -\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin n\pi t + \pi\cos n\pi t\right] \end{aligned}$$

To see whether this converges, we apply the test for divergence, which says that if a series converges the terms must go to zero as $n \to \infty$. The term $(1/n)\sin n\pi t$ does indeed go to zero, because of the $1/n$ factor. But the term $\cos n\pi t$ generally does not. If we plug in $t = 1/2$, we get the sequence $\cos n\pi/2$, which goes as $0, -1, 0, 1, 0, -1, 0, 1, \ldots$, and hence does not converge to zero.

(4) (5 points) Let $F(t)$ be the odd function of period 2π such that $F(t) = 1$ for $0 < t < \pi$ (this is a square wave). Consider the mass-spring system with $m = 1$, $k = 5$, subject to the driving force $F(t)$:

$$\frac{d^2x}{dt^2} + 5x = F(t)$$

Using Fourier series methods, find a steady periodic solution of this differential equation.

As we have seen several times in class, the square wave of amplitude 1 and period 2π is has a Fourier series

$$F(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin nt$$

We expect a steady periodic solution of the form

$$x_{\text{sp}}(t) = A_0/2 + \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt)$$

Plugging this into $x'' + 5x$ gives

$$x''_{\text{sp}} + 5x_{\text{sp}} = 5A_0/2 + \sum_{n=1}^{\infty} (A_n[5 - n^2] \cos nt + B_n[5 - n^2] \sin nt)$$

In order for this to equal $F(t)$, A_0 and A_n must equal zero, since $F(t)$ has no constant term or cosine terms. Also, B_n must be zero for n even, since $F(t)$ has no even sine terms. Thus we find

$$B_n[5 - n^2] = \frac{4}{\pi} \frac{1}{n} \quad (n \text{ odd})$$

$$B_n = \frac{4}{\pi} \frac{1}{n(5 - n^2)} \quad (n \text{ odd})$$

Note that this works because $5 - n^2$ is never zero ($\sqrt{5}$ is not an integer). The steady periodic solution is

$$x_{\text{sp}}(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n(5 - n^2)} \sin nt$$

References

[1] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, fourth edition.