MATH 285 E1/F1 GRADED HOMEWORK SET 4
DUE FRIDAY OCTOBER 24 IN LECTURE

This time, the homework has **just one part**. Please staple your homework together, and put your **name and section** on it. *Thank you!*

(1) (10 points) Find the *general solution* of the differential equation

$$(D-3)y = xe^{2x} + e^{3x}$$

Hint: Use the annihilator method.

An annihilator of e^{3x} is $D-3$. An annihilator of xe^{2x} is $(D-2)^2$. Thus an annihilator of their sum is $A(D) = (D-2)^2(D-3)$. Applying this to both sides of the equation, we get

$$(D-2)^2(D-3)(D-3)y = (D-2)^2(D-3)(xe^{2x} + e^{3x})$$
$$(D-2)^2(D-3)^2y = 0$$

This means that the function y we want to find must have the form

$$y = Ae^{2x} + Bxe^{2x} + Ce^{3x} + Exe^{3x}$$

We now plug this in to the original equation and see what coefficients are determined by that condition. To to this systematically, let's apply the operator $(D-3)$ to each of the terms

$$(D-3)(e^{2x}) = 2e^{2x} - 3e^{2x} = -e^{2x}$$
$$(D-3)(xe^{2x}) = 2xe^{2x} + e^{2x} - 3xe^{2x} = -xe^{2x} + e^{2x}$$
$$(D-3)(e^{3x}) = 3e^{3x} - 3e^{3x} = 0$$
$$(D-3)(xe^{3x}) = 3xe^{3x} + e^{3x} - 3xe^{3x} = e^{3x}$$

Putting it all together, we get

$$(D-3)y = -Ae^{2x} - Bxe^{2x} + Be^{2x} + Ee^{3x} = (B-A)e^{2x} - Bxe^{2x} + Ee^{3x}$$

Since this should be equal to $xe^{2x} + e^{3x}$, we must have $B - A = 0$, $-B = 1$, and $E = 1$, while C can be anything. Thus the general solution is

$$y = -e^{2x} - xe^{2x} + Ce^{3x} + xe^{3x}$$

(2) (5 points) Consider the forced mass-spring oscillator with mass $m = 4$, no damping $c = 0$, and a spring constant k that is adjustable. Consider a driving forces that is the sum of two cosine functions

$$F(t) = \cos 2t + \cos 5t.$$

The differential equation for the displacement $x(t)$ is then

$$4x'' + kx = \cos 2t + \cos 5t.$$

For what values of k is it impossible to find a particular solution of the form $A\cos 2t + B\cos 5t$? (These are the values of k for which resonance occurs.)

We try to find a solution of the form

$$x(t) = A\cos 2t + B\cos 5t$$

The second derivative is

$$x''(t) = -4A\cos 2t - 25B\cos 5t$$

Thus

$$4x'' + kx = (k-16)A\cos 2t + (k-100)B\cos 5t$$

Since this is to be equal to $\cos 2t + \cos 5t$, We must have

$$A = \frac{1}{k-16}, \quad B = \frac{1}{k-100}$$

At this step we have to assume that $k-16$ is not zero, and that $k-100$ is not zero. Conversely, this shows us the values of k for which resonance occurs: $k = 16$ and $k = 100$. For these values of k, the constants A and B cannot be found that make $A\cos 2t + B\cos 5t$ a solution of the equation.

(3) (5 points) Recall from the lecture the damped ($c > 0$) forced oscillator obeying the differential equation

$$mx'' + cx' + kx = F_0 \cos \omega t.$$

In the lecture, we derived a particular solution of this equation

$$x_p(t) = C\cos(\omega t - \alpha),$$

where

$$C = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}, \quad \tan\alpha = \frac{c\omega}{k - m\omega^2}.$$

Here is something we did not talk about in the lecture: the solution $x_p(t)$ is called the *steady-state* solution, because any solution will asymptotically converge to it as $t \to \infty$. More precisely, if $x_1(t)$ is *any solution* of the same nonhomogeneous equation, then

$$\lim_{t\to\infty} [x_1(t) - x_p(t)] = 0.$$

Explain why this is true. You should use the description of the solutions of the homogeneous equation $mx'' + cx' + kx = 0$ on page 191 of the textbook. *Hint:* It is very important that $c > 0$.

The first thing to say is that the difference $x_1(t) - x_p(t)$ is a solution of the homogeneous equation $mx'' + cx' + kx = 0$. This means that it falls into one of the three cases (underdamped, critically damped, or overdamped) shown on page 191. We want to show that in all three cases, the solutions of the homogeneous equation tend to zero as t goes to infinity.

Underdamped case: The general solution is

$$x(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)$$

The thing to recognize is that the factor e^{-pt} is going to zero. This is because p is an abbreviation for $c/2m$, which is positive, and therefore $\lim_{t\to\infty} e^{-pt} = 0$. Thus $\lim_{t\to\infty} x(t) = 0$ as well.

Critically damed case: The general solution is

$$x(t) = e^{-pt}(c_1 + c_2 t)$$

Again $p = c/2m$ is positive, so the factor e^{-pt} is exponentially decaying. However, this time there is a factor of t in the second term that goes to infinty. We just need to use L'Hopital's rule to see that the product actually goes to zero:

$$\lim_{t\to\infty} te^{-pt} = \lim_{t\to\infty} \frac{t}{e^{pt}} = \lim_{t\to\infty} \frac{1}{pe^{pt}} = 0$$

Overdamped case: The general solution is

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where r_1 and r_2 are the solutions of the characteristic equation

$$mr^2 + cr + k = 0$$

That is,

$$r_1, r_2 = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

We just need to see that both r_1 and r_2 are actually negative. The quantity

$$\frac{-c - \sqrt{c^2 - 4km}}{2m}$$

is clearly negative since both terms in the numerator are negative.

The other root

$$\frac{-c + \sqrt{c^2 - 4km}}{2m}$$

will be negative as long as $c > \sqrt{c^2 - 4km}$. But this is clear: $c^2 - 4km$ is less than c^2, so $\sqrt{c^2 - 4km}$ is less than c. This completes the proof.

(4) (5 points) Use the orthogonality properties of sine and cosine to compute the following integral

$$\int_{-\pi}^{\pi} (5\sin 2t - \cos 3t)(\cos 2t + 5\sin 6t + \sin 2t)\, dt$$

When we expand out the integrand, we get products of $\sin nt$ and $\cos mt$, namely,

$$5\sin 2t\cos 2t + 25\sin 2t\sin 6t + 5\sin 2t\sin 2t - \cos 3t\cos 2t - 5\cos 3t\sin 6t - \cos 3t\sin 2t$$

When this is integrated from $-\pi$ to π, the orthogonality properties say that a term such as $\cos mt\cos nt$ will integrate to zero unless $m = n$, that a term such as $\sin mt\sin nt$ will integrate to zero unless

$m = n$, and that a term such as $\cos mt \sin nt$ will always integrate to zero no matter what m and n are. If we look at what we have, the only term that gives a nonzero integral is $5 \sin 2t \sin 2t$. Thus the integral we were asked to compute simplifies to

$$\int_{-\pi}^{\pi} 5 \sin 2t \sin 2t \, dt = 5\pi$$