MATH 285 E1/F1 GRADED HOMEWORK SET 3
DUE WEDNESDAY OCTOBER 8 IN LECTURE

This time, the homework has **just one part**. Please staple your homework together, and put your **name and section** on it. *Thank you!*

(1) (5 points) Show that the functions e^{3x}, xe^{3x}, and x^2e^{3x} are linearly independent. That is, show that, if

$$c_1e^{3x} + c_2xe^{3x} + c_3x^2e^{3x} = 0 \text{ for all } x,$$

then it necessarily follows that $c_1 = c_2 = c_3 = 0$.

Hint: Get relations between the c_i's by plugging in $x = 0$ and/or differentiating the equation.

Plug in $x = 0$ to get

$$c_1e^0 + c_2 0e^0 + c_3 0e^0 = 0$$

Thus $c_1 = 0$. With that knowledge, we can differentiate to get

$$c_2(3xe^{3x} + e^{3x}) + c_3(3x^2e^{3x} + 2xe^{3x}) = 0$$

Plug in $x = 0$ to get

$$c_2(0 + e^0) + c_3(0 + 0) = 0$$

Thus $c_2 = 0$. With that knowledge, we can differentiate a second time to get

$$c_3(9x^2e^{3x} + 6xe^{3x} + 6xe^{3x} + 2e^{3x}) = 0$$

Plug in $x = 0$ to get

$$c_3(0 + 0 + 0 + 2e^0) = 0$$

Thus $c_3 = 0$, and we are done.

(2) (15 points) Consider the polynomial

$$\begin{aligned} p(r) = r^{12} &- 12r^{11} + 51r^{10} + r^9 - 1968r^8 + 19003r^7 \\ &- 106948r^6 + 440432r^5 - 1423168r^4 + 3448064r^3 \\ &- 6069248r^2 + 7606272r - 5160960. \end{aligned}$$

Find the general solution of the linear homogeneous equation

$$p(D)y = 0,$$

where as usual $D = \frac{d}{dx}$, and $p(D)$ denotes the constant coefficient differential operator obtained by substituting D for r in $p(r)$. We are looking for a general solution that is real (not complex) and which involves twelve arbitrary constants.

Hint: You should use the following factorization of $p(r)$:

$$p(r) = (r-3)^2(r-4)^3(r+7)(r^2-r+5)(r^2+16)^2$$

The given factorization of $p(r)$ gives a factorizaton of $p(D)$:

$$p(D) = (D-3)^2(D-4)^3(D+7)(D^2-D+5)(D^2+16)^2$$

The general solution of $p(D)y = 0$ is a combination of the general solutions for each of the factors. We proceed factor by factor:

(a) $(D-3)^2y = 0$. There is a real double root at 3. The basic solutions are e^{3x} and xe^{3x}. The general solution is
$$y = c_1e^{3x} + c_2xe^{3x}.$$

(b) $(D-4)^3y = 0$. There is a real triple root at 4. The basic solutions are e^{4x}, xe^{4x}, and x^2e^{4x}. The general solution is
$$y = c_3e^{4x} + c_4xe^{4x} + c_5x^2e^{4x}.$$

(c) $(D+7)y = 0$. There is a real simple root at -7. The basic solution is e^{-7x}. The general solution is
$$y = c_6e^{-7x}.$$

(d) $(D^2 - D + 5)y = 0$. There are complex roots at
$$\frac{1}{2} \pm i\frac{\sqrt{19}}{2}$$
There is a complex solution
$$e^{[(1/2)+i(\sqrt{19}/2)]x} = e^{(1/2)x}e^{i(\sqrt{19}/2)x} = e^{(1/2)x}\left[\cos\left(\frac{\sqrt{19}}{2}x\right) + i\sin\left(\frac{\sqrt{19}}{2}x\right)\right]$$
The real and imaginary parts of this solution are also solutions, so we have basic solutions
$$e^{(1/2)x}\cos\left(\frac{\sqrt{19}}{2}x\right), \quad e^{(1/2)x}\sin\left(\frac{\sqrt{19}}{2}x\right).$$
The general solution is
$$y = c_7e^{(1/2)x}\cos\left(\frac{\sqrt{19}}{2}x\right) + c_8e^{(1/2)x}\sin\left(\frac{\sqrt{19}}{2}x\right)$$

(e) $(D^2+16)^2$. The characteristic polynomial $(r^2+16)^2$ factors into complex factors as
$$(r^2+16)^2 = [(r-4i)(r+4i)]^2 = (r-4i)^2(r+4i)^2$$
Thus there are double roots at $\pm 4i$. Thus there are complex solutions e^{4ix} and e^{-4ix}, and additionally xe^{4ix} and xe^{-4ix}. The real and imaginary parts of the solution
$$e^{4ix} = \cos(4x) + i\sin(4x)$$

namely, $\cos(4x)$ and $\sin(4x)$, are also solutions of the equation, as are the real and imaginary parts of

$$xe^{4ix} = x\cos(4x) + ix\sin(4x)$$

Thus the four basic real solutions are $\cos(4x)$, $\sin(4x)$, $x\cos(4x)$, and $x\sin(4x)$. The general solution is

$$y = c_9\cos(4x) + c_{10}\sin(4x) + c_{11}x\cos(4x) + c_{12}x\sin(4x).$$

Putting it all together, the general solution of the equation $p(D)y = 0$ is

$$\begin{aligned} y &= c_1e^{3x} + c_2xe^{3x} + c_3e^{4x} + c_4xe^{4x} \\ &+ c_5x^2e^{4x} + c_6e^{-7x} + c_7e^{(1/2)x}\cos\left(\frac{\sqrt{19}}{2}x\right) + c_8e^{(1/2)x}\sin\left(\frac{\sqrt{19}}{2}x\right) \\ &+ c_9\cos(4x) + c_{10}\sin(4x) + c_{11}x\cos(4x) + c_{12}x\sin(4x). \end{aligned}$$

Or in a slightly more compact form:

$$\begin{aligned} y &= (c_1 + c_2x)e^{3x} + (c_3 + c_4x + c_5x^2)e^{4x} \\ &+ c_6e^{-7x} + e^{(1/2)x}\left[c_7\cos\left(\frac{\sqrt{19}}{2}x\right) + c_8\sin\left(\frac{\sqrt{19}}{2}x\right)\right] \\ &+ c_9\cos(4x) + c_{10}\sin(4x) + x[c_{11}\cos(4x) + c_{12}\sin(4x)]. \end{aligned}$$

(3) (10 points) A body of mass $m = 5$ kg is attached to a spring with spring constant $k = 20$ kg/s^2. The body is suspended on an "air-hockey" table so that it is not subject to friction or gravity, but it is connected to a dashpot mechanism that lets us adjust the degree of damping, $c \geq 0$. The equation of motion is the usual one

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

Consider the following experimental procedure: move the body to an initial position $x(0) = 0$ m, and set it into motion with initial velocity $v(0) = 1$ m/s.

Your task is to determine the position function $x(t)$ in the following three cases:
(a) $c = 10$ kg/s,
(b) $c = 20$ kg/s,
(c) $c = 30$ kg/s.
It is perfectly fine to refer to section 3.4 of the textbook in your solution (that is, you don't have to rederive everything all over again, but you should mention what facts from the textbook you are using).

We will use SI units througout. The characteristic equation is $5r^2 + cr + 20 = 0$. Its discriminant is $c^2 - 4(5)(20) = c^2 - 400$. The nature of the solutions depends on whether $c^2 < 400$, $c^2 = 400$, or

$c^2 > 400$. The first is underdamped, the second is critically damped, and the third is overdamped.

(a) $c = 10$. Thus $c^2 - 400 = -300 < 0$, so this is underdamped. The roots of the characteristic equation are

$$r = \frac{-10 \pm \sqrt{-300}}{10} = -1 \pm i\sqrt{3}$$

The general solution is given by equation (21) in section 3.4:

$$x(t) = e^{-t}(A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t))$$

Using the initial condition $x(0) = 0$, we get

$$0 = e^0(A\cos(0) + B\sin(0)) = A$$

Thus $A = 0$. Differentiating the solution we get

$$v(t) = -e^{-t}B\sin(\sqrt{3}t) + \sqrt{3}e^{-t}B\cos(\sqrt{3}t)$$

Using the initial condition $v(0) = 1$, we get

$$1 = \sqrt{3}B$$

or $B = 1/\sqrt{3}$. So the solution is

$$x(t) = \frac{1}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t)$$

(b) $c = 20$. Thus $c^2 - 400 = 0$, so this is critically damped. The characteristic equation has just one root

$$r = \frac{-20}{10} = -2$$

The general solution is given by equation (20) in section 3.4:

$$x(t) = e^{-2t}(c_1 + c_2 t)$$

Using the initial condition $x(0) = 0$, we get

$$0 = e^0(c_1 + c_2 0)$$

Thus $c_1 = 0$. Differentiating the solution we get

$$v(t) = -2e^{-2t}(c_2 t) + e^{-2t}c_2$$

Using $v(0) = 1$, we get

$$1 = 0 + e^0 c_2$$

Thus $c_2 = 1$. The solution is

$$x(t) = te^{-2t}$$

(c) $c = 30$. Thus $c^2 - 400 = 500 > 0$, so this is overdamped. The roots of the characteristic equation are

$$r = \frac{-30 \pm \sqrt{500}}{10} = -3 \pm \sqrt{5}$$

The general solution is given by equation (19) in section 3.4:

$$x(t) = c_1 e^{(-3+\sqrt{5})t} + c_2 e^{(-3-\sqrt{5})t}$$

Using $x(0) = 0$, we obtain

$$0 = c_1 e^0 + c_2 e^0$$

or $c_1 + c_2 = 0$. Differentiating the general solution, we get

$$v(t) = (-3 + \sqrt{5})c_1 e^{(-3+\sqrt{5})t} + (-3 - \sqrt{5})c_2 e^{(-3-\sqrt{5})t}$$

Using $v(0) = 1$, we get

$$1 = (-3 + \sqrt{5})c_1 + (-3 - \sqrt{5})c_2$$

The equation $c_1 + c_2 = 0$ means we can write $c_2 = -c_1$, so the second equation becomes

$$1 = (-3 + \sqrt{5})c_1 - (-3 - \sqrt{5})c_1 = 2\sqrt{5}c_1$$

Thus $c_1 = 1/2\sqrt{5}$, and $c_2 = -1/2\sqrt{5}$. So the solution is

$$x(t) = (1/2\sqrt{5})[e^{(-3+\sqrt{5})t} - e^{(-3-\sqrt{5})t}]$$

One may also observe that this is equal to

$$x(t) = \frac{1}{\sqrt{5}} e^{-3t} \sinh(\sqrt{5}t)$$

(4) (5 points) Find a particular solution of

$$y'' - 4y' + y = 3e^x.$$

We use the method of undetermined coefficients, so we try a solution of the form $y = Ae^x$. Thus $y' = Ae^x$ and $y'' = Ae^x$. Plugging this into the equation,

$$Ae^x - 4Ae^x + Ae^x = 3e^x$$
$$-2Ae^x = 3e^x$$
$$A = -3/2$$

So $y = (-3/2)e^x$ is a particular solution.