MATH 285 E1/F1 GRADED HOMEWORK SET 2
DUE FRIDAY SEPTEMBER 26 IN LECTURE

LET'S CHANGE IT UP WITH SOME FRESH NEW INSTRUCTIONS: This time, the homework has **two parts, A and B**. Please turn in each part separately, with your **name and section** clearly marked on each part. Please staple all the pages for a particular part together, but do not staple the two parts to each other. When you turn the homework in, there will be two boxes. *Thank you!*

Part A

(1) (5 points) Consider the equation

$$(2y + xe^{xy})\frac{dy}{dx} + x + ye^{xy} = 0$$

Show that this equation is exact, and find an implicit equation for the solution.

The equation has the form $M(x,y)+N(x,y)\frac{dy}{dx} = 0$ where $M(x,y) = x+ye^{xy}$ and $N(x,y) = 2y+xe^{xy}$. We compare the partial derivatives

$$\frac{\partial M}{\partial y} = 0 + e^{xy} + xye^{xy},$$

$$\frac{\partial N}{\partial x} = 0 + e^{xy} + xye^{xy}$$

Since these are the same, the equation is exact.

Now we look for a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = x + ye^{xy}, \quad \frac{\partial F}{\partial y} = 2y + xe^{xy}.$$

Integrating the equation for the x-derivative gives

$$F(x,y) = \int (x + ye^{xy})\, dx = x^2/2 + e^{xy} + C(y),$$

where $C(y)$ is some function of y only. Plugging this into the equation for the y-derivative gives

$$2y + xe^{xy} = \frac{\partial}{\partial y}\left(x^2/2 + e^{xy} + C(y)\right) = xe^{xy} + C'(y).$$

Thus $C'(y) = 2y$, and we can take $C(y) = y^2$. Thus $F(x,y) = x^2/2 + e^{xy} + y^2$ works. The implicit equation for the solutions is then

$$x^2/2 + e^{xy} + y^2 = C,$$

where C is an arbitrary constant.

(2) (5 points) Let $P(t)$ denote the number of technology startups in the San Francisco Bay area, where time t is measured in months. Startups are created when two recent graduates decide to found one, and they either fail or they are bought by Google. The rate at which startups are formed is given by $(10-0.01P)P$ per month. Also, each month, 10% of the startups fail, and 5% are bought by Google. How many startups do you expect to exist at a particular time many months into the future? That is, what is $\lim_{t\to\infty} P(t)$?

The differential equation for $P(t)$ is

$$\frac{dP}{dt} = (10 - 0.01P)P - 0.1P - 0.05P,$$

where the first term is the number of startups formed per month, the term $0.1P$ is the number of startups that fail per month, and $0.05P$ is the number of startups that are bought by Google per month. We can write this as a logistic equation:

$$\frac{dP}{dt} = (9.85 - 0.01P)P = 0.01P(985 - P)$$

We know that for the logistic equation $\frac{dP}{dt} = kP(M-P)$, the value of P converges to M in the long run. So in this case, the number of startups will converge to 985 in the long run, that is, $\lim_{t\to\infty} P(t) = 985$.

Part B

(3) (5 points) Suppose that $y_1(x)$ and $y_2(x)$ are two solutions of the *nonhomogeneous* equation

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

Prove that their difference $y_1(x) - y_2(x)$ is a solution of the *homogeneous* equation

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

Let us abbreviate $A(x)$ as just A, and so on. We need to consider the expression

$$Z = A(y_1 - y_2)'' + B(y_1 - y_2)' + C(y_1 - y_2)$$

and show that it is zero. First expand out the terms, using the facts

$$(y_1 - y_2)' = y_1' - y_2', \quad (y_1 - y_2)'' = y_1'' - y_2''$$

to get

$$Z = Ay_1'' - Ay_2'' + By_1' - By_2' + Cy_1 - Cy_2.$$

Then collect the terms involving y_1 and y_2 to get

$$Z = (Ay_1'' + By_1' + Cy_1) - (Ay_2'' + By_2' + Cy_2).$$

Now use the fact that y_1 and y_2 each solve the nonhomogeneous equation:

$$Ay_1'' + By_1' + Cy_1 = F, \quad Ay_2'' + By_2' + Cy_2 = F.$$

Thus

$$Z = F - F = 0,$$

and we are done.

(4) (15 points)

(a) Find the general solution of the second order linear homogeneous equation

$$4y'' + 8y' + 3y = 0.$$

The characteristic equation is

$$4r^2 + 8r + 3 = 0$$

The quadratic formula yields

$$r = (1/8)(-8 \pm \sqrt{64 - 48}) = (1/8)(-8 \pm 4) = -1/2 \text{ or } -3/2$$

Thus the general solution is

$$y(x) = c_1 e^{(-1/2)x} + c_2 e^{(-3/2)x}$$

(b) Using your innate cleverness, find a particular solution to the nonhomongeneous equation

$$4y'' + 8y' + 3y = 15.$$

The clever thing is to try a very simple function, in fact, a constant. Plugging in $y(x) = C$, we need to satisfy

$$4(0) + 8(0) + 3C = 15$$

since the first and second derivatives of a constant are zero. Thus $y(x) = 5$ is a particular solution.

(c) Using the results of the previous two parts, find solution of the initial value problem

$$4y'' + 8y' + 3y = 15, \quad y(0) = 0, \quad y'(0) = 0.$$

The general solution of the nonhomogeneous problem is the general solution of the homogeneous problem plus a particular solution, so

$$y(x) = c_1 e^{(-1/2)x} + c_2 e^{(-3/2)x} + 5$$

Pluggin in the initial condition for $y(0)$, we get

$$0 = y(0) = c_1 + c_2 + 5$$

Taking the derivative of the general solution, we get

$$y'(x) = (-1/2)c_1 e^{(-1/2)x} + (-3/2)c_2 e^{(-3/2)x}$$

Plugging in initial condition:

$$0 = y'(0) = (-1/2)c_1 + (-3/2)c_2$$

From the second equation, we get $c_1 = -3c_2$, which we plug into the first to get $0 = -3c_2 + c_2 + 5$. Thus $c_2 = 5/2$, and $c_1 = -15/2$. So the solution to the initial value problem is

$$y(x) = (-15/2)e^{(-1/2)x} + (5/2)e^{(-3/2)x} + 5$$