NAME: Solutions

NetID:

MATH 285 E1/F1 Exam 3 (B) **November 14, 2014** **Instructor: Pascaleff**

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- The exam is 50 minutes.
- Do not discuss this exam with anyone until after 3:00 pm on Nov. 14, 2014.

Orthogonality formulas

$$\int_{-L}^{L} \cos\frac{m\pi t}{L}\cos\frac{n\pi t}{L}\,dt = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{1}$$

$$\int_{-L}^{L} \sin\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{2}$$

$$\int_{-L}^{L} \cos\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt = 0 \tag{3}$$

Some integral formulas

$$\int u\cos u\,du = u\sin u + \cos u + C \tag{4}$$

$$\int u\sin u\,du = -u\cos u + \sin u + C \tag{5}$$

1. (20 points) Find the general solution of the differential equation

$$y' - 5y = xe^{5x}$$

$$(D-5)y = xe^{5x}$$

Annihilator of xe^{5x} is $(D-5)^2$

Apply to both sides $(D-5)^3 y = 0$

so try $y = Ae^{5x} + Bxe^{5x} + Cx^2e^{5x}$

$$y' = 5Ae^{5x} + B(e^{5x} + 5xe^{5x}) + C(2xe^{5x} + 5x^2e^{5x})$$

$$-5y = -5Ae^{5x} - 5Bxe^{5x} - 5Cx^2e^{5x}$$

$$y' - 5y = Be^{5x} + 2Cxe^{5x}$$

Must match xe^{5x}

So $A =$ anything $B = 0$ $2C = 1 \Rightarrow C = \frac{1}{2}$

general solution $\boxed{y = Ae^{5x} + \frac{1}{2}x^2e^{5x}}$ A any constant

2. (20 points) Consider the forced oscillator with mass $m = 1$, spring constant $k = 5$, no damping $c = 0$, and forcing function $F(t)$:

$$F(t) = \cos 2t + \sin 4t + 3 \sin 6t$$

Find a particular solution of the differential equation $mx'' + kx = F(t)$.

$x'' + 5x = \cos 2t + \sin 4t + 3\sin 6t$

Try $x = A_1 \cos 2t + B_1 \sin 2t + A_2 \cos 4t + B_2 \sin 4t$
$+ A_3 \cos 6t + B_3 \sin 6t$

$x'' = -4A_1 \cos 2t - 4B_1 \sin 2t - 16A_2 \cos 4t - 16B_2 \sin 4t$
$- 36A_3 \cos 6t - 36 B_3 \sin 6t$

$x'' + 5x = A_1 \cos 2t + B_1 \sin 2t - 11A_2 \cos 4t - 11B_2 \sin 4t$
$- 31 A_3 \cos 6t - 31 B_3 \sin 6t$

must match $\cos 2t + \sin 4t + 3\sin 6t$

$A_1 = 1 \quad B_1 = 0 \quad A_2 = 0 \quad -11B_2 = 1 \quad A_3 = 0 \quad -31B_3 = 3$

$B_2 = \frac{-1}{11} \quad B_3 = \frac{-3}{31}$

$\therefore \boxed{x(t) = \cos 2t - \frac{1}{11} \sin 4t - \frac{3}{31} \sin 6t}$

3. (a) (10 points) Suppose that a function $f(t)$ which is periodic of period 2π has the Fourier series

$$f(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \sin nt$$

Use the orthogonality formulas to evaluate the integral

$$\int_{-\pi}^{\pi} f(t) \sin 3t \, dt$$

$$\int_{-\pi}^{\pi} \sum \frac{(-1)^n}{n^2+1} \sin nt \sin 3t \, dt = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \int_{-\pi}^{\pi} \sin nt \sin 3t \, dt$$

All the terms are zero except when $n=3$:

$$= \frac{(-1)^3}{3^2+1} \int_{-\pi}^{\pi} \sin 3t \sin 3t \, dt = \frac{-1}{10} \pi = \boxed{\frac{-\pi}{10}}$$

(b) (10 points) Let $g(t)$ be the function which is periodic of period 20, and which is defined on the interval $-10 \le t < 10$ by the formula

$$g(t) = 7t^2 + 2t + 3$$

Set up, but do not evaluate, an integral expression for the coefficient of $\sin \frac{3\pi t}{10}$ in the Fourier series of $g(t)$ (also known as b_3 in our standard notation).

$$\boxed{b_3 = \frac{1}{10} \int_{-10}^{10} (7t^2 + 2t + 3) \sin \frac{3\pi t}{10} \, dt}$$

4. (a) (5 points) Consider the function which is periodic of period 2π defined on the interval $-\pi \le t < \pi$

$$f(t) = \begin{cases} t, & -\pi \le t < 0 \\ 609250, & t = 0 \\ 8, & 0 < t < \pi \end{cases}$$

If we take the Fourier series of $f(t)$, and put $t = 0$ in that series, what number does it converge to? Put another way, what is the sum of the Fourier series of $f(t)$ at $t = 0$? Explain your answer (briefly).

$\lim_{t \to 0^-} f(t) = 0 \qquad \lim_{t \to 0^+} f(t) = 8$

Since $t = 0$ is a point of discontinuity, the Fourier series converges to the average of the one-sided limits. $\frac{1}{2}[0 + 8] = \boxed{4}$

(b) (15 points) Consider the function defined by the Fourier series

$$g(t) = \sum_{n=1}^{\infty} 4e^{-4n} \sin n\pi t$$

Find a Fourier series expression for the antiderivative $\int g(t)\, dt$. You are *not* expected to address the question of convergence.

$$\int g(t)\,dt = \sum_{n=1}^{\infty} 4e^{-4n} \int \sin n\pi t \, dt$$

$$= \sum_{n=1}^{\infty} 4e^{-4n} \frac{-1}{n\pi} \cos n\pi t + C$$

$$= \sum_{n=1}^{\infty} \frac{-4e^{-4n}}{n\pi} \cos n\pi t + C$$

5. (20 points) Find the Fourier series of the periodic function of period 2 defined on the interval $-1 \le t < 1$ by

$$f(t) = 4|t|, \quad -1 \le t < 1$$

Hint: You should use the fact that $f(t)$ is an even function.

$f(t)$ even $\Rightarrow$ sine cofficients $b_n = 0$ for all n

$$a_0 = \frac{2}{1}\int_0^1 f(t)\,dt = 2\int_0^1 4t\,dt = 4\left[t^2\right]_0^1 = 4(1-0) = 4$$

$$a_n = \frac{2}{1}\int_0^1 f(t)\cos\frac{n\pi t}{1}\,dt = 2\int_0^1 4t\cos n\pi t\,dt$$

$\{$let $u = n\pi t \quad du = n\pi\,dt\}$

$$= 8\left(\frac{1}{n\pi}\right)^2 \int_0^1 (n\pi t)(\cos n\pi t)\,n\pi\,dt = 8\left(\frac{1}{n\pi}\right)^2 \int_0^{n\pi} u\cos u\,du$$

$$= 8\left(\frac{1}{n\pi}\right)^2 \Big[u\sin u + \cos u\Big]_0^{n\pi} \quad \text{by formula}$$

$$= 8\left(\frac{1}{n\pi}\right)^2 \left[n\pi\sin n\pi + \cos n\pi - 0\sin 0 - \cos 0\right]$$

$$= 8\left(\frac{1}{n\pi}\right)^2 \left[\cos n\pi - 1\right]$$

$$= 8\left(\frac{1}{n\pi}\right)^2 \left[(-1)^n - 1\right] = 8\left(\frac{1}{n\pi}\right)^2 \begin{Bmatrix} -2 & n \text{ odd} \\ 0 & n \text{ even} \end{Bmatrix}$$

$$= \begin{Bmatrix} \dfrac{-16}{(n\pi)^2} & n \text{ odd} \\ 0 & n \text{ even} \end{Bmatrix}$$

continues

This page is for work that doesn't fit on other pages.

Thus the Fourier series of $f(t)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t$$

$$= \frac{4}{2} + \sum_{n=1}^{\infty} \begin{Bmatrix} \frac{-16}{(n\pi)^2} & n \text{ odd} \\ 0 & n \text{ even} \end{Bmatrix} \cos n\pi t$$

$$= \boxed{2 + \sum_{n \text{ odd}} \frac{-16}{(n\pi)^2} \cos n\pi t}$$