NAME: Solutions

NetID:

MATH 285 E1/F1 Exam 3 (A) **November 14, 2014** **Instructor: Pascaleff**

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- The exam is 50 minutes.
- Do not discuss this exam with anyone until after 3:00 pm on Nov. 14, 2014.

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Orthogonality formulas

$$\int_{-L}^{L} \cos\frac{m\pi t}{L}\cos\frac{n\pi t}{L}\,dt = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{1}$$

$$\int_{-L}^{L} \sin\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{2}$$

$$\int_{-L}^{L} \cos\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt = 0 \tag{3}$$

Some integral formulas

$$\int u\cos u\,du = u\sin u + \cos u + C \tag{4}$$

$$\int u\sin u\,du = -u\cos u + \sin u + C \tag{5}$$

1. (20 points) Find the general solution of the differential equation

$$y' - 3y = xe^{3x}$$

$$(D-3)y = xe^{3x}$$

Annihilator of xe^{3x} is $(D-3)^2$

Apply to both sides $(D-3)^3 y = 0$

so try $y = Ae^{3x} + Bxe^{3x} + Cx^2e^{3x}$

$$y' = 3Ae^{3x} + B(e^{3x} + 3xe^{3x}) + C(2xe^{3x} + 3x^2e^{3x})$$

$$-3y = -3Ae^{3x} - 3Bxe^{3x} - 3Cx^2e^{3x}$$

$$y' - 3y = Be^{3x} + 2Cxe^{3x}$$

Must match xe^{3x}

So $A =$ anything $B = 0$ $2C = 1 \Rightarrow C = \frac{1}{2}$

general solution $\boxed{y = Ae^{3x} + \frac{1}{2}x^2e^{3x}}$ A any constant

2. (20 points) Consider the forced oscillator with mass $m = 1$, spring constant $k = 10$, no damping $c = 0$, and forcing function $F(t)$:

$$F(t) = \sin 2t + 2\sin 4t + \cos 6t$$

Find a particular solution of the differential equation $mx'' + kx = F(t)$.

$$x'' + 10x = \sin 2t + 2\sin 4t + \cos 6t$$

Try $x = A_1\cos 2t + B_1 \sin 2t + A_2 \cos 4t + B_2 \sin 4t$
$+ A_3 \cos 6t + B_3 \sin 6t$

$x'' = -4A_1 \cos 2t - 4B_1 \sin 2t - 16A_2 \cos 4t - 16B_2 \sin 4t$
$- 36A_3 \cos 6t - 36 B_3 \sin 6t$

$x'' + 10x = 6A_1 \cos 2t + 6B_1 \sin 2t - 6A_2 \cos 4t - 6B_2 \sin 4t$
$- 26A_3 \cos 6t - 26 B_3 \sin 6t$

must match $\sin 2t + 2\sin 4t + \cos 6t$

$A_1 = 0$ $\quad 6B_1 = 1$ $\quad A_2 = 0$ $\quad -6B_2 = 2$ $\quad -26A_3 = 1$ $\quad B_3 = 0$

$B_1 = \frac{1}{6}$ $\quad B_2 = \frac{-1}{3}$ $\quad A_3 = \frac{-1}{26}$

$$\therefore \boxed{x(t) = \frac{1}{6}\sin 2t - \frac{1}{3}\sin 4t - \frac{1}{26}\cos 6t}$$

3. (a) (10 points) Suppose that a function $f(t)$ which is periodic of period 2π has the Fourier series

$$f(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \sin nt$$

Use the orthogonality formulas to evaluate the integral

$$\int_{-\pi}^{\pi} f(t) \sin 4t \, dt$$

$$\int_{-\pi}^{\pi} \sum \frac{(-1)^{n+1}}{n(n+1)} \sin nt \sin 4t \, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \int_{-\pi}^{\pi} \sin nt \sin 4t \, dt$$

All the terms are zero except when $n=4$:

$$= \frac{(-1)^{4+1}}{4(4+1)} \int_{-\pi}^{\pi} \sin 4t \sin 4t \, dt = \frac{(-1)^5}{4 \cdot 5} \pi = \boxed{\frac{-\pi}{20}}$$

(b) (10 points) Let $g(t)$ be the function which is periodic of period 30, and which is defined on the interval $-15 \leq t < 15$ by the formula

$$g(t) = 2 + 3t + 6t^2$$

Set up, but do not evaluate, an integral expression for the coefficient of $\cos \frac{3\pi t}{15}$ in the Fourier series of $g(t)$ (also known as a_3 in our standard notation).

$$\boxed{a_3 = \frac{1}{15} \int_{-15}^{15} (2 + 3t + 6t^2) \cos \frac{3\pi t}{15} \, dt}$$

4. (a) (5 points) Consider the function which is periodic of period 2π defined on the interval $-\pi \le t < \pi$

$$f(t) = \begin{cases} 16, & -\pi \le t < 0 \\ 609250, & t = 0 \\ t, & 0 < t < \pi \end{cases}$$

If we take the Fourier series of $f(t)$, and put $t = 0$ in that series, what number does it converge to? Put another way, what is the sum of the Fourier series of $f(t)$ at $t = 0$? Explain your answer (briefly).

$\lim_{t \to 0^-} f(t) = 16 \qquad \lim_{t \to 0^+} f(t) = 0$

Since $t = 0$ is a point of discontinuity, the Fourier series converges to the average of the one-sided limits. $\frac{1}{2}[16 + 0] = \boxed{8}$

(b) (15 points) Consider the function defined by the Fourier series

$$g(t) = \sum_{n=1}^{\infty} 3e^{-2n} \sin n\pi t$$

Find a Fourier series expression for the antiderivative $\int g(t)\,dt$. You are *not* expected to address the question of convergence.

$$\int g(t)\,dt = \sum_{n=1}^{\infty} 3e^{-2n} \int \sin n\pi t\,dt$$

$$= \sum_{n=1}^{\infty} 3e^{-2n} \frac{-1}{n\pi} \cos n\pi t + C$$

$$= \sum_{n=1}^{\infty} \frac{-3e^{-2n}}{n\pi} \cos n\pi t + C$$

5. (20 points) Find the Fourier series of the periodic function of period 2 defined on the interval $-1 \le t < 1$ by

$$f(t) = 2|t|, \quad -1 \le t < 1$$

Hint: You should use the fact that $f(t)$ is an even function.

$f(t)$ even $\Rightarrow$ sine cofficients $b_n = 0$ for all n

$$a_0 = \frac{2}{1}\int_0^1 f(t)\,dt = 2\int_0^1 2t\,dt = 2\left[t^2\right]_0^1 = 2(1-0) = 2$$

$$a_n = \frac{2}{1}\int_0^1 f(t)\cos\frac{n\pi t}{1}\,dt = 2\int_0^1 2t\cos n\pi t\,dt$$

$$\left\{\text{let } u = n\pi t \quad du = n\pi\,dt\right\}$$

$$= 4\left(\frac{1}{n\pi}\right)^2\int_0^1 (n\pi t)(\cos n\pi t)\,n\pi\,dt = 4\left(\frac{1}{n\pi}\right)^2\int_0^{n\pi} u\cos u\,du$$

$$= 4\left(\frac{1}{n\pi}\right)^2\left[u\sin u + \cos u\right]_0^{n\pi} \quad \text{by formula}$$

$$= 4\left(\frac{1}{n\pi}\right)^2\left[n\pi\sin n\pi + \cos n\pi - 0\sin 0 - \cos 0\right]$$

$$= 4\left(\frac{1}{n\pi}\right)^2\left[\cos n\pi - 1\right]$$

$$= 4\left(\frac{1}{n\pi}\right)^2\left[(-1)^n - 1\right] = 4\left(\frac{1}{n\pi}\right)^2\begin{cases} -2 & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$$

$$= \begin{cases} \dfrac{-8}{(n\pi)^2} & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$$

continues

This page is for work that doesn't fit on other pages.

Thus the Fourier series of $f(t)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t$$

$$= \frac{2}{2} + \sum_{n=1}^{\infty} \begin{Bmatrix} \frac{-8}{(n\pi)^2} & n \text{ odd} \\ 0 & n \text{ even} \end{Bmatrix} \cos n\pi t$$

$$= \boxed{1 + \sum_{n \text{ odd}} \frac{-8}{(n\pi)^2} \cos n\pi t}$$