NAME: Solutions

NetID:

MATH 285 E1/F1 Exam 2 (B) **October 17, 2014** **Instructor: Pascaleff**

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- The exam is 50 minutes.
- Do not discuss this exam with anyone until after 3:00 pm on Oct. 17, 2014.

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20 points) Let $P(t)$ denote the population of a penguin colony in Antarctica. We assume that each female lays one egg each year, so the birth rate is 0.5 births per penguin per year. Due to scare resources, the death rate depends on the population as $.01 + .0001P$ deaths per penguin per year.

   Suppose that the penguin population is in *equilibrium*, meaning that it is constant in time: $P(t) = P_0$. What are the possible values of the equilibrium (constant) population P_0?

$$\frac{dP}{dt} = .5P - (.01 + .0001P)P$$

$$\frac{dP}{dt} = .49P - .0001P^2$$

If population is constant, $P(t) = P_0$, then $\frac{dP}{dt} = 0$

$$0 = \frac{dP}{dt} = .49P_0 - .0001P_0^2$$

$$0 = P_0(.49 - .0001P_0)$$

$$\Rightarrow P_0 = 0 \quad \text{or} \quad .49 - .0001P_0 = 0$$

$$.49 = .0001P_0$$

$$4900 = P_0$$

So $\boxed{P_0 = 0 \quad \text{or} \quad P_0 = 4900}$

2. (20 points) Show that the functions $f(x) = 2e^{x^2}$ and $g(x) = e^{5x}$ and $h(x) = e^{1+x^2}$ are *not* linearly independent. That is, find constants A, B, C such that

$$Af(x) + Bg(x) + Ch(x) = 0 \text{ for all } x$$

$h(x) = e^{1+x^2} = e^1 e^{x^2}$

So $e f(x) - 2h(x) = e 2 e^{x^2} - 2 e e^{x^2} = 0$

$$\boxed{\begin{gathered} A = e, \ B = 0, \ C = -2 \\ e f(x) + 0 g(x) - 2 h(x) = 0 \end{gathered}}$$

3. (20 points) For each polynomial differential operator $p(D)$, find the solutions to the homogeneous differential equation $p(D)y = 0$, where $D = \frac{d}{dx}$. It is not necessary to rederive the solution completely, but keep in mind that partial credit can be given if substantial work is shown. In the first three parts, you are asked to find the general real (not complex) solution. In the last part, you are asked to find one complex solution.

(a) $p(D) = D^2 - 4D + 3$. Find the general real solution of $p(D)y = 0$.

$r^2 - 4r + 3 = 0$

$(r-1)(r-3) = 0$

$r_1 = 1 \quad r_2 = 3$

$y(x) = c_1 e^{x} + c_2 e^{3x}$

(b) $p(D) = (D-3)^3(D+2)$. Find the general real solution of $p(D)y = 0$.

roots basic solutions

$r_1 = 3 \quad e^{3x},\ xe^{3x},\ x^2e^{3x}$

$r_2 = -2 \quad e^{-2x}$

$y(x) = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x} + c_4 e^{-2x}$

(c) $p(D) = D^2 + 3D + 5$. Find the general real solution of $p(D)y = 0$.

$$r^2 + 3r + 5 \qquad r = \frac{-3 \pm \sqrt{9 - 4\cdot 5}}{2} = \frac{-3 \pm i\sqrt{11}}{2}$$

Basic solutions $\quad e^{-\frac{3}{2}x}\cos\left(\frac{\sqrt{11}}{2}x\right), \; e^{-\frac{3}{2}x}\sin\left(\frac{\sqrt{11}}{2}x\right)$

$$y(x) = c_1 e^{-\frac{3}{2}x}\cos\left(\frac{\sqrt{11}}{2}x\right) + c_2 e^{-\frac{3}{2}x}\sin\left(\frac{\sqrt{11}}{2}x\right)$$

(d) $p(D) = D - 5i$, where $i = \sqrt{-1}$. In this case, there are no real solutions. Find one complex solution of $p(D)y = 0$.

$$y(x) = e^{5ix}$$

4. (20 points) A mass is attached to a spring and a dashpot, so that its position $x(t)$ obeys the differential equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

The mass m, damping coefficient c, and spring constant k are given by $m = 4, c = 8, k = 4$.

Suppose we do an experiment where the initial position and the initial velocity are $x(0) = -1$ and $v(0) = 10$. The function $x(t)$ is determined by these initial conditions plus the differential equation. How many times does the mass pass through the equilibrium position $x = 0$? That is, how many positive numbers t are there such that $x(t) = 0$? Note that if the solutions oscillate the answer could be infinitely many.

$$4x'' + 8x' + 4x = 0$$

$$4r^2 + 8r + 4 = 0 \qquad r = \frac{-8 \pm \sqrt{8^2 - 4\cdot 4\cdot 4}}{2\cdot 4} = -1 \pm 0$$

repeated root : $x(t) = e^{-t}(c_1 + c_2 t)$

$$-1 = x(0) = e^0(c_1 + 0) = c_1$$

$$x(t) = e^{-t}(-1 + c_2 t)$$

$$v(t) = x'(t) = e^{-t}c_2 + (-e^{-t})(-1 + c_2 t)$$

$$10 = e^0 c_2 + e^0(1)$$

$$10 = c_2 + 1 \qquad c_2 = 9$$

$$x(t) = e^{-t}(-1 + 9t)$$

When is $x(t) = 0$? $\qquad e^{-t}(-1 + 9t) = 0$

$$-1 + 9t = 0$$

$$t = 1/9 > 0$$

$\therefore$ the mass passes through equilibrium $\boxed{\text{once}}$.

5. (20 points) Find the *general solution* of the differential equation

$$y'' + 5y' + 6y = 3x$$

Try $y = Ax + B$

$y' = A$

$y'' = 0$

$y'' + 5y' + 6y = 5A + 6(Ax+B) = 6Ax + 5A + 6B$

want $6Ax + 5A + 6B = 3x$

So $6A = 3 \longrightarrow A = 1/2$

$5A + 6B = 0 \longrightarrow \frac{5}{2} + 6B = 0 \longrightarrow B = \frac{-5}{12}$

So $y_p = \frac{1}{2}x - \frac{5}{12}$

Need to add solution of $y'' + 5y' + 6y = 0$

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$y_c = c_1 e^{-2x} + c_2 e^{-3x}$

$y(x) = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{2}x - \frac{5}{12}$