NAME: Solutions

NetID:

MATH 285 E1/F1 Exam 1 (A) **September 19, 2014** **Instructor: Pascaleff**

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

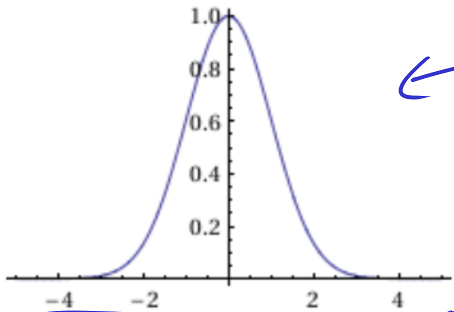
INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.

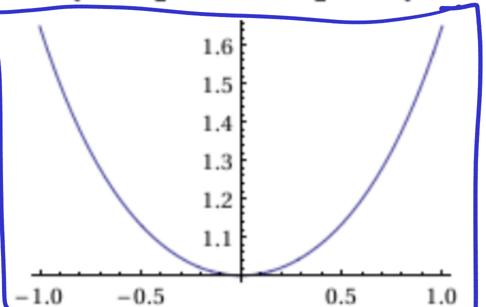
1. (20 points) Consider the differential equation

$$\frac{dy}{dx} = xy$$

Which of the following graphs could be a solution curve of this equation? Circle all that apply.



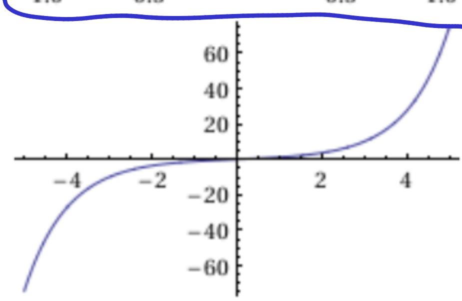
No, since slope in Quadrant I is negative, whereas $xy > 0$



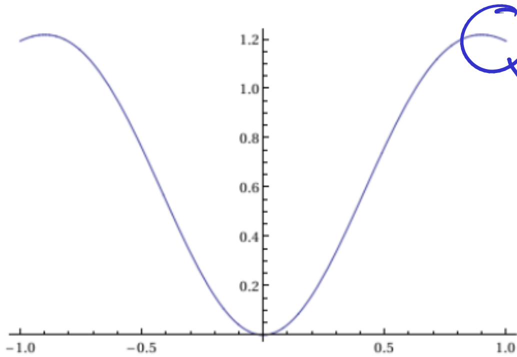
Yes, in fact $y(x) = Ce^{\frac{1}{2}x^2}$ is the general solution.

$\left[\frac{dy}{dx} - xy = 0 \quad \rho = e^{\int -x\,dx} = e^{-\frac{1}{2}x^2} \right.$

$\left. \frac{d}{dx}\left(e^{-\frac{1}{2}x^2} y\right) = 0 \quad e^{-\frac{1}{2}x^2} y = C \right]$



No, since $y \equiv 0$ is a solution, other solutions can't cross x-axis



No, slope is negative here, whereas $xy > 0$ there

2. (20 points) An object moves along a one-dimensional axis. Its motion is descibed by a function $x(t)$. It is subjected to an acceleration given by

$$a(t) = 1 + \pi \sin(\pi t).$$

Suppose that at $t = 0$, the velocity is zero: $v(0) = 0$. What is the net change in position between $t = 0$ and $t = 1$? That is, what is $x(1) - x(0)$?

$$\frac{dv}{dt} = a(t) = 1 + \pi \sin(\pi t)$$

$$v = \int a(t)\,dt = \int (1 + \pi \sin \pi t)\,dt = t - \cos(\pi t) + C$$

Find C:

$$0 = v(0) = 0 - \cos(\pi 0) + C = 0 - 1 + C$$

$$1 = C$$

So $v(t) = t - \cos(\pi t) + 1$

Integrate $v(t)$ from 0 to 1:

$$x(1) - x(0) = \int_0^1 (t - \cos(\pi t) + 1)\,dt$$

$$= \left[\frac{1}{2}t^2 - \frac{\sin(\pi t)}{\pi} + t\right]_0^1$$

$$= \frac{1}{2} - \frac{\sin(\pi)}{\pi} + 1 - 0 + \frac{\sin(0)}{\pi} - 0$$

$$= \frac{1}{2} + 1 = \boxed{\frac{3}{2}}$$

3. (20 points) Find the general solution, valid for $x > 0$, of

$$\frac{dy}{dx} = \frac{x^4 + 2y}{x}$$

Hint: Linear equation, integrating factor.

Write in standard linear form

$$\frac{dy}{dx} = x^3 + \frac{2y}{x} \longrightarrow \frac{dy}{dx} - \frac{2}{x}y = x^3$$

$$P(x) = \frac{-2}{x} \quad Q(x) = x^3$$

Integrating factor:

$$\rho(x) = e^{\int P(x)dx} = e^{\int \frac{-2}{x}dx} = e^{-2\ln(x)}$$

$$= \left(e^{\ln(x)}\right)^{-2} = x^{-2}$$

Multiply: $x^{-2}\frac{dy}{dx} - 2x^{-1}y = x$

Product rule $\frac{d}{dx}\left(x^{-2}y\right) = x$

Integrate: $x^{-2}y = \frac{1}{2}x^2 + C$

$$\boxed{y = \frac{1}{2}x^4 + Cx^2}$$

4. (20 points) Consider the equation

$$\frac{dy}{dx} - \frac{2}{x}y = y^2$$

Use the substitution $u = y^{-1}$ to transform this equation into a linear equation for u. Do not solve the resulting equation; the purpose of this problem is merely to transform the original equation for y into one for u.

$$u = y^{-1} \rightarrow y = u^{-1} \rightarrow \frac{dy}{dx} = -u^{-2}\frac{du}{dx}$$

$$\frac{dy}{dx} - \frac{2}{x}y = y^2$$

$$-u^{-2}\frac{du}{dx} - \frac{2}{x}u^{-1} = u^{-2}$$

mult. by $-u^2$

$$\boxed{\frac{du}{dx} + \frac{2}{x}u = -1}$$

5. (20 points) A metal ball has been heated to $1000°C$. It is placed into a bath of ice water at $0°C$. After 5 seconds, it has cooled to a temperature of $(1000e^{-10})°C$ (approximately $0.045°C$).

Suppose now that the metal ball is heated again to $1000°C$, but instead it is placed into boiling water at $100°C$. How long will it take to reach a temperature of $200°C$?

In both situation, the cooling process is governed by Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - A)$$

where A is the temperature of the water, and k is a constant.

First, solve N.L.C. using separation of variables

$$\int \frac{dT}{T-A} = \int -k\,dt \rightarrow \ln|T-A| = -kt + C$$

$$\rightarrow \quad T - A = \pm e^{C} e^{-kt} = D e^{-kt}$$

so $T = A + De^{-kt}$.

First situation: $A = 0$ $\quad T(0) = 1000$

$T(5) = 1000e^{-10}$

So $\quad 1000 = T(0) = 0 + De^{-k0} = D$

Thus $D = 1000$

$$1000e^{-10} = T(5) = 0 + 1000e^{-k\cdot 5}$$

$$e^{-10} = e^{-k\cdot 5}$$

$$-10 = -k\cdot 5$$

$$2 = k$$

Thus $k = 2$.

Solution continues $\longrightarrow$

This page is for work that doesn't fit on the other pages. Please indicate the problem that the work goes with.

Second situation: $A = 100$ $T(0) = 1000$

$$1000 = T(0) = 100 + De^{-2\cdot 0}$$

$$1000 = 100 + D$$

Thus $D = 900$.

$$T(t) = 100 + 900e^{-2t}$$

When is $T(t) = 200$?

$$200 = 100 + 900e^{-2t}$$

$$100 = 900e^{-2t}$$

$$\frac{1}{9} = e^{-2t}$$

$$\ln\left(\frac{1}{9}\right) = -2t$$

$$\boxed{t = -\frac{1}{2}\ln\left(\frac{1}{9}\right)}$$

this can be further simplified to $\ln(3)$, but that is not necessary.