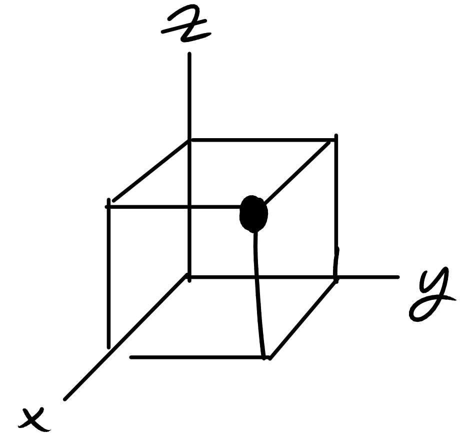
This lecture is not covered on the final exam.

No more Homework

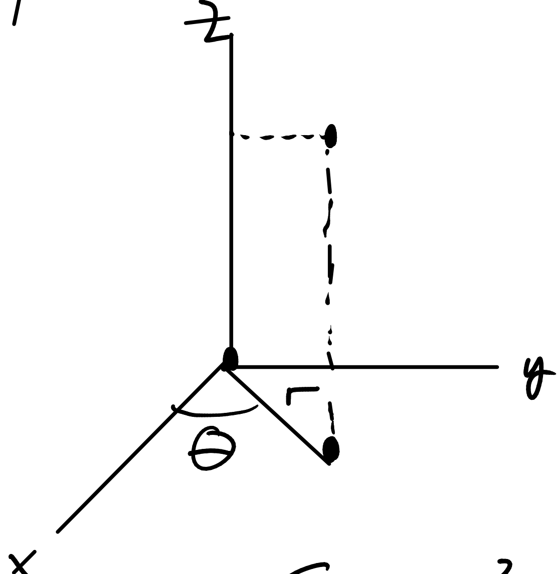
Post some review problem (not for credit)

3d coordinate systems, change of coordinates in 3 dimensions.

Rectangular coordinates



Cylindrical coordinates
" polar coordinates with an extra z direction



project to xy plane and use polar coords there

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

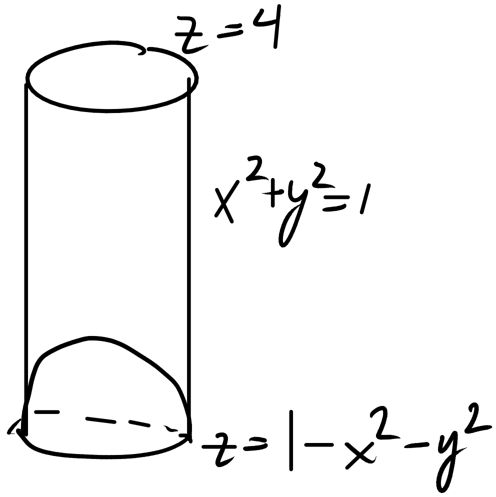
$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \\ z = z \end{cases}$$

Integral in cylindrical coordinates

$$\iiint f(x,y,z)\,dV = \iiint f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

$$dV = \underbrace{dA}_{xy}\,dz$$

<u>Example</u>

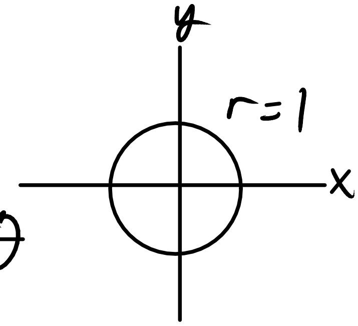


density is proportional to $r=\sqrt{x^2+y^2}$

density $= Kr$

find mass.

$$\text{Mass} = \iiint Kr\cdot r\,dr\,d\theta\,dz$$



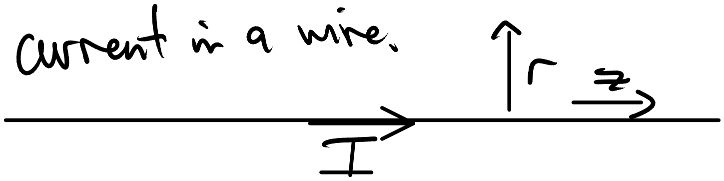
$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=1-r^2}^{z=4} (Kr^2)\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} Kr^2(4-(1-r^2))\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 Kr^2(3+r^2)\,dr\,d\theta = K\int_0^{2\pi}\int_0^1 (3r^2+r^4)\,dr\,d\theta$$

$$= K\int_0^{2\pi}\left[r^3+\frac{1}{5}r^5\right]_0^1 d\theta = K\int_0^{2\pi}\frac{6}{5}\,d\theta = \frac{12\pi K}{5}$$

current in a wire.



magnetic field strength depends on distance to wire.

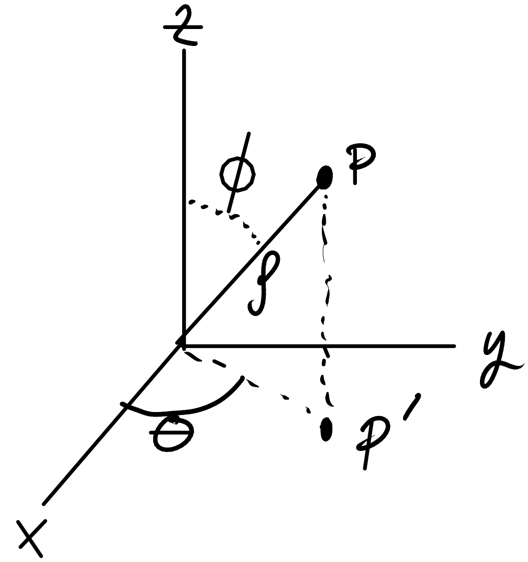
Spherical coordinates ρ = distance from $(0,0,0)$

two angles θ, ϕ

ϕ = angle between z-axis and line thru O and P

P' = projection of P onto xy-plane

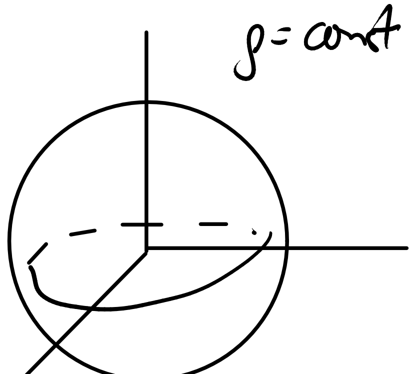
θ = polar coord for P'



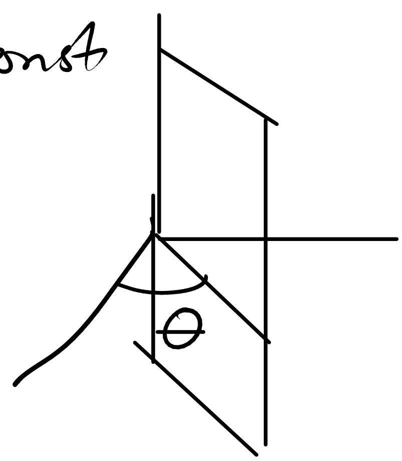
Conventions $\rho \geq 0$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$

$\rho = \text{const}$

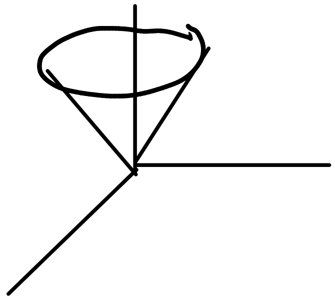
Sphere centered at O



$\theta = \text{const}$

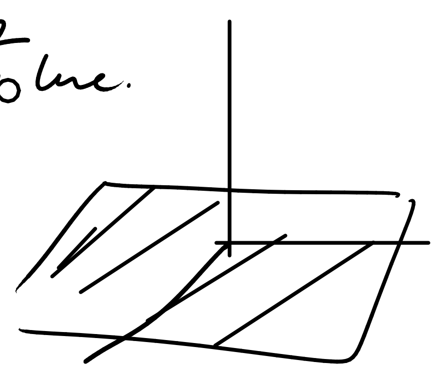


$\phi = \text{const}$

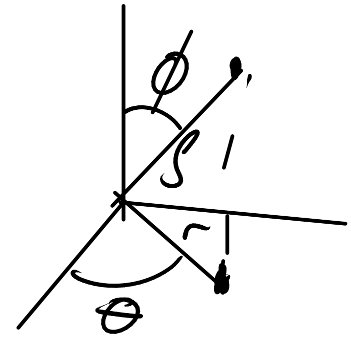


$\phi = \pi/2$

xy-plane.



spherical → cylindrical



ρ $\qquad\qquad$ $r = \rho \sin \phi$

θ $\qquad\qquad$ $\theta = \theta$

ϕ $\qquad\qquad$ $z = \rho \cos \phi$

cylindrical → rectangular

r $\qquad\qquad$ $x = r\cos\theta$

θ $\qquad\qquad$ $y = r\sin\theta$

z $\qquad\qquad$ $z = z$

$$\left\{ \begin{array}{l} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{array} \right\}$$

$$\rho^2 = x^2 + y^2 + z^2$$

ρ = "radius" $\qquad$ ϕ = zenith angle

θ = azimuth angle.

Triple integral

$$\iiint_E f(x,y,z)\,dV = \iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$dV = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

Volume of sphere of radius $R = \frac{4}{3}\pi R^3$

$$V = \iiint 1\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} 1\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{3}\rho^3\right]_0^R \sin\phi \, d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\frac{1}{3}R^3 \sin\phi \, d\theta \, d\phi = \int_0^{\pi}\frac{2\pi}{3}R^3 \sin\phi \, d\phi$$

$$= \frac{2\pi}{3}R^3\int_0^{\pi}\sin\phi \, d\phi = \frac{2\pi}{3}R^3\left[-\cos\phi\right]_0^{\pi}$$

$$= \frac{2\pi}{3}R^3\left[-(-1)-(1)\right] = \frac{4\pi}{3}R^3$$

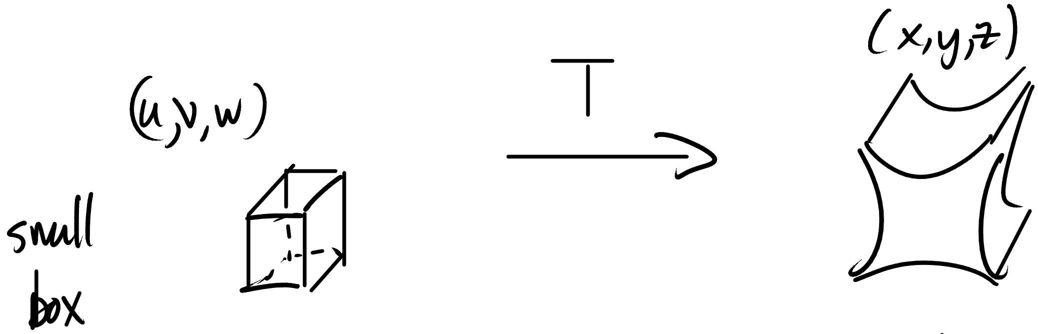
Change of variables in 3d

Jacobian factor that measures how the change of coordinates distorts volumes.
given by determinant of matrix of partial derivatives.

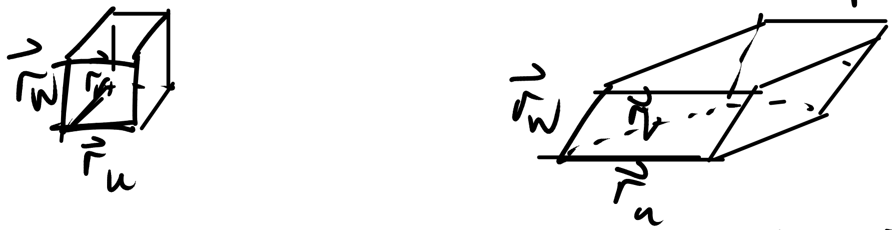
$$(x,y,z) = T(u,v,w) \qquad \begin{aligned} x &= g(u,v,w) \\ y &= h(u,v,w) \\ z &= h(u,v,w) \end{aligned}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{array}{c} \left|\begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{array}\right| \\ \begin{array}{ccc} \vec{r}_u & \vec{r}_v & \vec{r}_w \end{array} \end{array}$$

3x3 Jacobian.



infinitesimal box $\longrightarrow$ infinitesimal parallelepiped



Jacobian = volume of this infinitesimal parallelepiped.

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$\begin{vmatrix} \sin\phi \cos\theta & \rho \cos\phi \cos\theta & -\rho \sin\phi \sin\theta \\ \sin\phi \sin\theta & \rho \cos\phi \sin\theta & \rho \sin\phi \cos\theta \\ \cos\phi & -\rho \sin\phi & 0 \end{vmatrix}$$

$$= \sin\phi \cos\theta \left(- -\rho \sin\phi \, \rho \sin\phi \cos\theta \right)$$
$$- \rho \cos\phi \cos\theta \left(- \cos\phi \, \rho \sin\phi \cos\theta \right)$$
$$+ -\rho \sin\phi \sin\theta \left(\sin\phi \sin\theta (-\rho \sin\phi) - \rho \cos\phi \sin\theta \cos\phi \right)$$
$$= \rho^2 \sin^3\phi \cos^2\theta + \rho^2 \cos^2\phi \sin\phi \cos^2\theta$$
$$+ \rho^2 \sin^3\phi \sin^2\theta + \rho^2 \sin\phi \cos^2\phi \sin^2\theta$$

$$= \rho^2 \sin\phi \left(\sin^2\phi \cos^2\theta + \cos^2\phi \cos^2\theta \right.$$
$$\left. \sin^2\phi \sin^2\theta + \cos^2\phi \sin^2\theta \right)$$

$$= \rho^2 \sin\phi \left(\cos^2\theta(\sin^2\phi + \cos^2\phi) + \sin^2\theta(\sin^2\phi + \cos^2\phi) \right)$$

$$= \rho^2 \sin\phi \left(\cos^2\theta + \sin^2\theta \right)$$

$$= \rho^2 \sin\phi$$

lots of other coordinates systems...