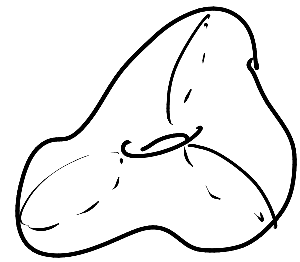
Triple integrals

Applications: Volume $E$



$$\text{Vol}(E) = \iiint_E dV$$

Could have variable density $\rho(x,y,z)$ $\left[\frac{\text{mass}}{\text{volume}}\right]$

$$\text{Mass} = \iiint_E \rho(x,y,z)\,dV$$

$$\boxed{\begin{array}{c}\text{constant } \rho: \\ M = \rho V\end{array}}$$

Center of mass $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \iiint_E x\,\rho(x,y,z)\,dV \Big/ \iiint_E \rho(x,y,z)\,dV$$

$$\bar{y} = \iiint_E y\,\rho(x,y,z)\,dV \Big/ \text{Total mass}$$

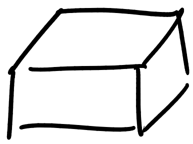
$$\bar{z} = \iiint_E z\,\rho(x,y,z)\,dV \Big/ \text{Total mass}$$

Case where density $\rho = 1$ constant

$$\bar{x} = \iiint_E x\,dV \Big/ \text{Vol}(E) \qquad \bar{y} = \iiint_E y\,dV \Big/ \text{Vol}(E)$$

$$\bar{z} = \iiint_E z\,dV \Big/ \text{Vol}(E)$$

"centroid"



$$\{a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$$

$$\int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx$$

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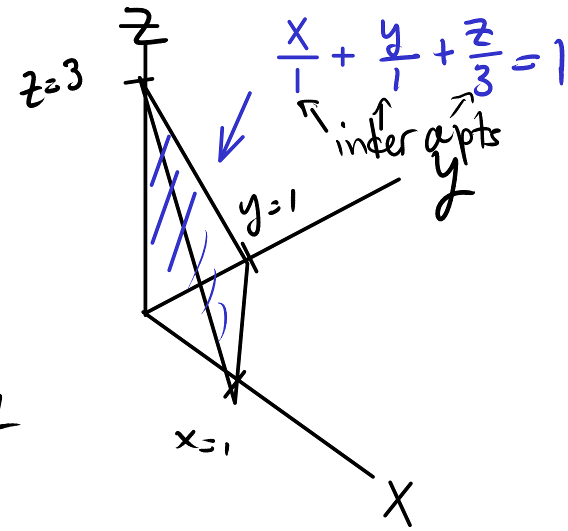
$$I = \iiint_E 2e^{3(x+y)+z}\,dV$$



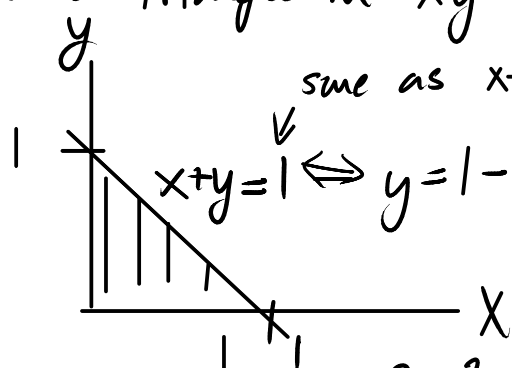

E = tetrahedron

slanted plane $3x + 3y + z = 3$

Top $\quad z = 3 - 3x - 3y$

Bottom $\quad z = 0$

E sits on a triangle in xy-plane




same as $x + y + \frac{z}{3} = 1$ when $z = 0$

$x + y = 1 \Leftrightarrow y = 1 - x$

$$\iiint_E f\,dV = \int_0^1 \int_0^{1-x} \int_0^{3-3x-3y} f\,dz\,dy\,dx$$

$$I = \int_0^1 \int_0^{1-x} \int_0^{3-3x-3y} 2e^{3(x+y)+z}\,dz\,dy\,dx$$

$$\left[\, 2e^{3(x+y)+z} = 2e^{3(x+y)} e^z \xrightarrow{\int} 2e^{3(x+y)} e^z \,\right]$$

$$I = \int_0^1 \int_0^{1-x} \left[ 2e^{3(x+y)+z} \right]_0^{3-3x-3y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[ 2e^{3(x+y)+3-3x-3y} - 2e^{3(x+y)} \right] dy\,dx$$

$$I = \int_0^1 \int_0^{1-x} \left( 2e^3 - 2e^{3(x+y)} \right) dy\,dx$$

$$e^{3(x+y)} = e^{3x+3y} = e^{3x} e^{3y} \xrightarrow{\int dy} e^{3x} \frac{1}{3} e^{3y}$$

$$I = \int_0^1 \left[ 2e^3 y - \frac{2}{3} e^{3(x+y)} \right]_0^{1-x} dx$$

$$= \int_0^1 \left( 2e^3(1-x) - \frac{2}{3} e^3 - 2e^3 \cdot 0 + \frac{2}{3} e^{3x} \right) dx$$

$$= \int_0^1 \left( \frac{4}{3} e^3 - 2e^3 x + \frac{2}{3} e^{3x} \right) dx$$

$$= \left[ \frac{4}{3} e^3 x - e^3 x^2 + \frac{2}{9} e^{3x} \right]_0^1$$

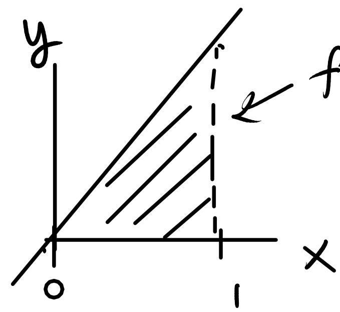
$$= \frac{4}{3} e^3 - e^3 + \frac{2}{9} e^3 - \frac{2}{9} \qquad \frac{4}{3} - 1 + \frac{2}{9} = \frac{5}{9}$$

$$= \frac{5}{9} e^3 - \frac{2}{9}$$

$$I = \iiint_E y\sin(\pi x^4)\,dV$$

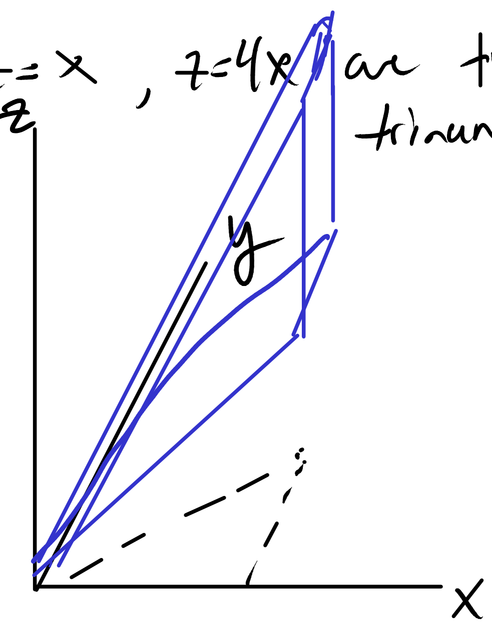
$$E = \{0 \le x \le 1,\ 0 \le y \le 3x,\ x \le z \le 4x\}$$

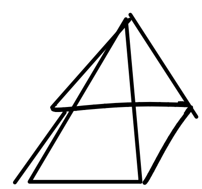
Draw it:




first two inequalities.

$z = x$, $z = 4x$ are two graphs living over this triangle



$$\{0 \le x \le 1,\ 0 \le y \le 3x,\ x \le z \le 4x\}$$

$$I = \int_0^1 \int_0^{3x} \int_x^{4x} y\sin(\pi x^4)\,dz\,dy\,dx$$

Wrong $\displaystyle\int_x^{4x}\int_0^{3x}\int_0^1 y\sin(\pi x^4)\,dx\,dy\,dz$

$$I = \int_0^1 \int_0^{3x} \int_x^{4x} y \sin(\pi x^4)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{3x} \Big[ y \sin(\pi x^4)\, z \Big]_x^{4x} dy\, dx$$

$$= \int_0^1 \int_0^{3x} \Big[ 3xy \sin(\pi x^4) \Big] dy\, dx$$

$$= \int_0^1 \Big[ 3x \frac{y^2}{2} \sin(\pi x^4) \Big]_0^{3x} dx$$

$$= \int_0^1 \Big[ 3x \frac{(3x)^2}{2} \sin(\pi x^4) \Big] dx = \frac{27}{2} \int_0^1 x^3 \sin(\pi x^4)\, dx$$

$u = \pi x^4 \quad du = 4\pi x^3 dx$

$$= \frac{27}{2} \int_0^{\pi} \sin u \frac{du}{4\pi} = \frac{27}{8\pi} \Big[ -\cos u \Big]_0^{\pi} = +\frac{27}{4\pi}$$

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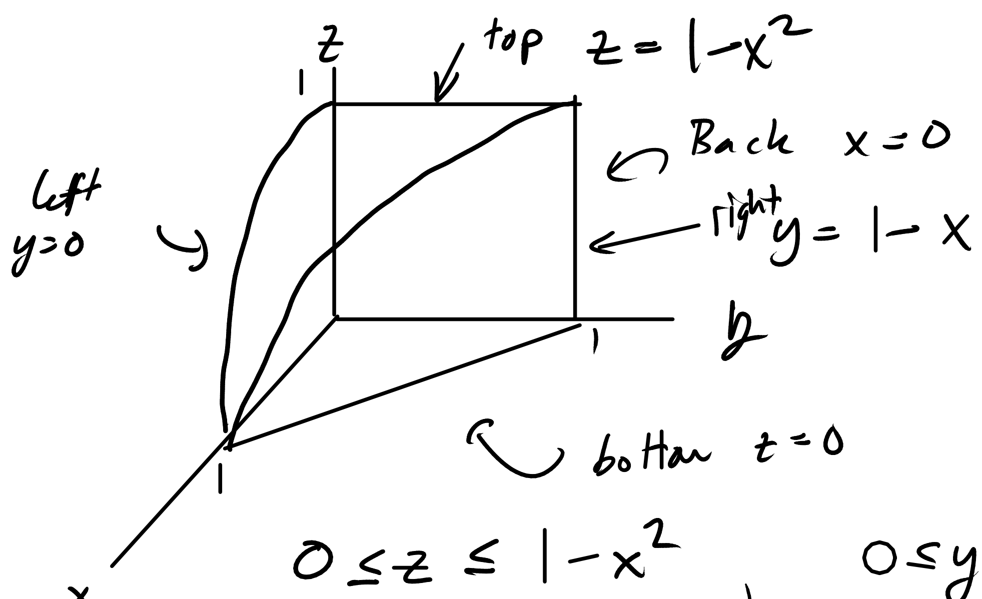



set up

$$\iiint_E f\, dV$$

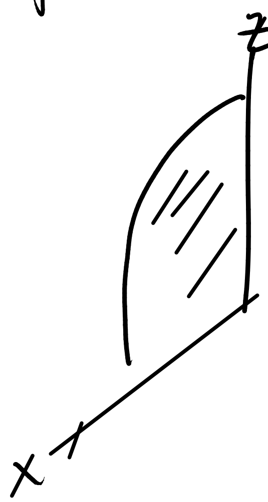
in all 6 possible orders.

$0 \le z \le 1 - x^2, \quad 0 \le y \le 1 - x, \quad 0 \le x \le 1$

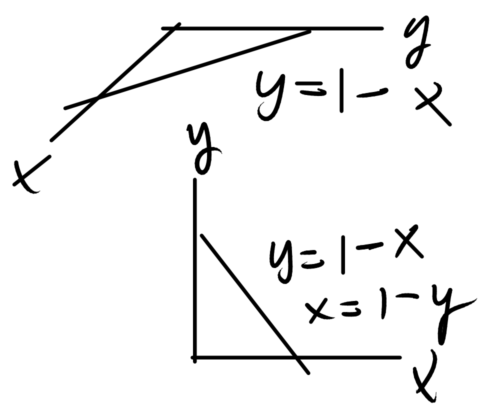
$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\, dz\, dy\, dx \qquad (zyx)$$

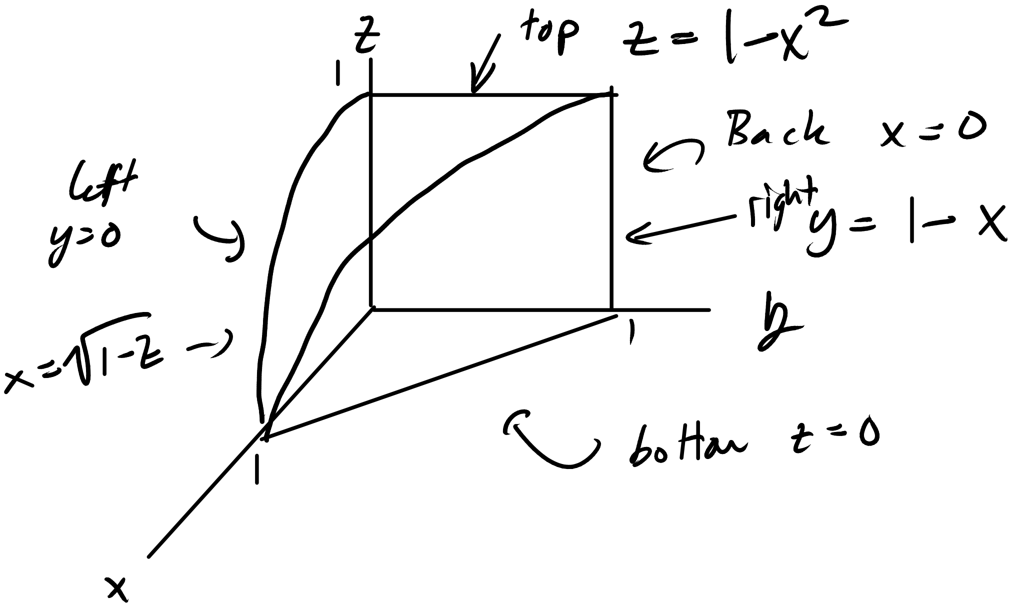
$$I = \int_0^1 \int_0^{1-x^2} \int_0^{1-x} f \quad dy\, \underbrace{dz\, dx} \qquad (yzx)$$

integrate over $y$ then integrate over region in $xz$ plane

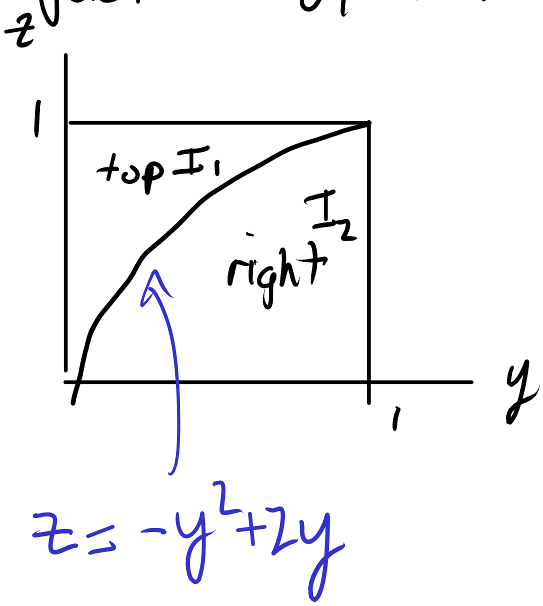



$$\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f \, dz\, \underbrace{dx\, dy} \qquad (zxy)$$






Project onto square in $yz$ plane




dividing curve corresponds to intersection of

$$\left.\begin{aligned} z &= 1 - x^2 \\ y &= 1 - x \end{aligned}\right\}$$

$$z = 1 - x^2 = 1 - (1-y)^2$$

$$z = 1 - (1 - 2y + y^2)$$
$$= -y^2 + 2y$$

Break up integral into $I_1$ $I_2$

$$I = \iiint_E f\,dV = I_1 + I_2$$

$$I_1 = \int_0^1 \int_{-y^2+2y}^{1} \int_0^{\sqrt{1-z}} f \quad dx\,dz\,dy$$

$$I_2 = \int_0^1 \int_0^{-y^2+2y} \int_0^{1-y} f \; dx\,dz\,dy$$