Change of variables cont'd, Triple integrals

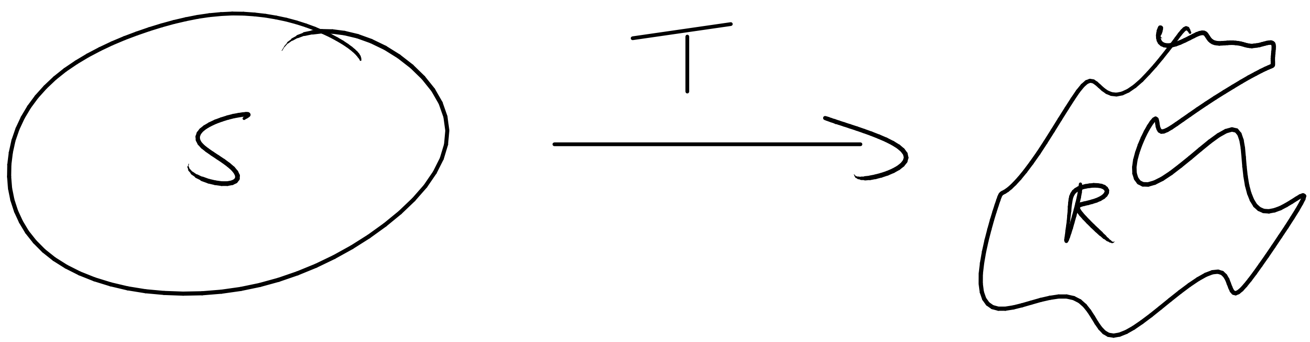
* Course instructor survey *

Justification for change of variables formula

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \qquad \underbrace{(x,y) = T(u,v)}_{\text{transformation}}$$

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

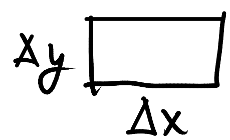
- R is the image of S under T
- $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$ Jacobian



Q: Why is there the Jacobian factor?

in (x,y) coordinates $\iint f(x,y)\,dA$

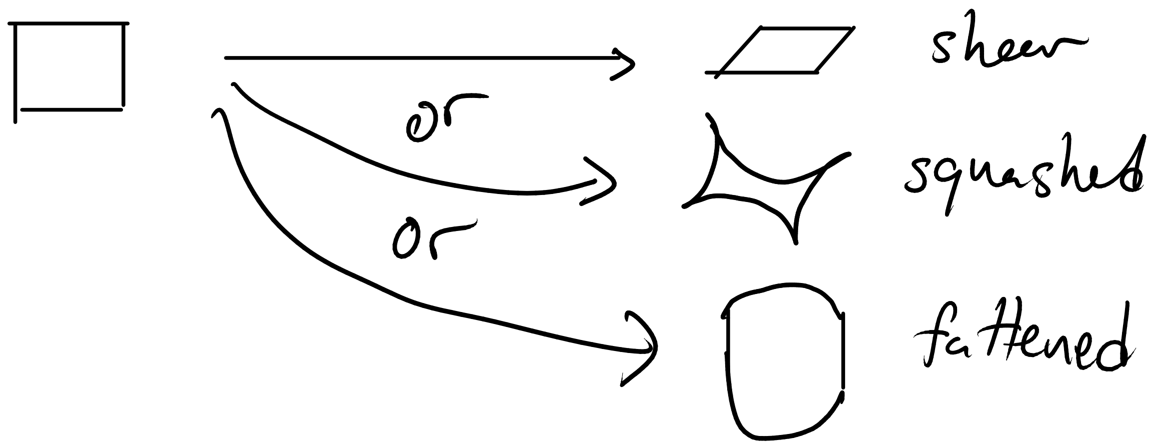
$$dA = dx\,dy$$



$$\Delta A = \Delta x \, \Delta y$$

Q: What happens to a small rectangle under Transformation T?

$$(u,v) \xrightarrow{T} (x,y)$$



this means that the transformation T may distort areas.

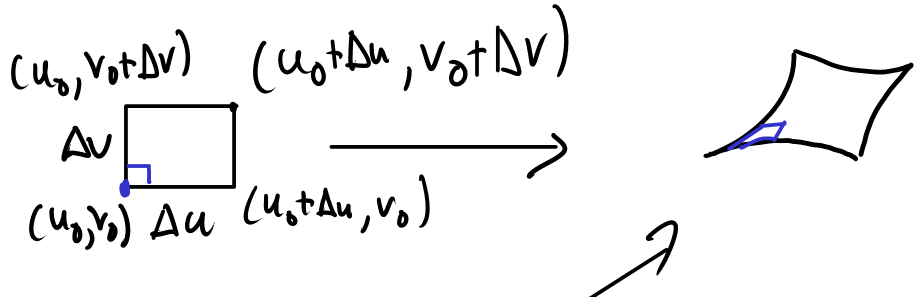
Areas don't match in the two coordinate systems.

Area in $(u,v) = du\,dv$ Area in $(x,y) = dx\,dy$

in general $du\,dv \neq dx\,dy$

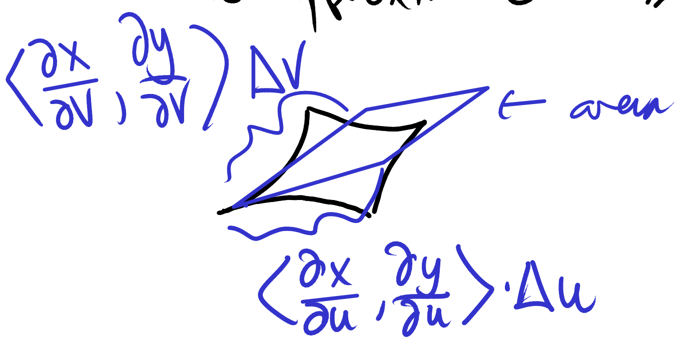
Want: express (x,y)-area $dx\,dy$ in (u,v) coordinates

$$dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \quad \text{in fact.}$$



← estimate area of this shape

let's approximate by a parallelogram.



look at lower side $v = v_0$

$T(u, v_0)$ describes the lower side in (x,y)-coordinates

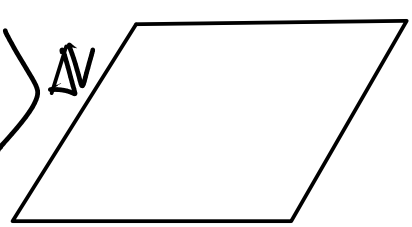
Approximate side as straight
Linear approximation on $T(u, v_0)$

$$T(u, v_0) \approx T(u_0, v_0) + \underbrace{\frac{\partial T}{\partial u}(u_0, v_0)}_{\left\langle \frac{\partial x}{\partial u}(u_0,v_0),\ \frac{\partial y}{\partial u}(u_0,v_0) \right\rangle} \Delta u$$

$T(u_0, v)$ describes ~~the~~ the "left side".

$$T(u_0, v) \approx T(u_0, v_0) + \underbrace{\frac{\partial T}{\partial v}(u_0, v_0)}_{\left\langle \frac{\partial x}{\partial v}(u_0,v_0),\ \frac{\partial y}{\partial v}(u_0,v_0) \right\rangle} \Delta v$$

$\vec{r}_v$
$= \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle \Delta v$



$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle \Delta u$

$$\text{Area} = \left| \vec{r}_u \times \vec{r}_v \right|$$

$$= \left| \left\langle \frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u, 0 \right\rangle \times \left\langle \frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v, 0 \right\rangle \right|$$

$$\text{Area} = \left|\vec{r}_u \times \vec{r}_v\right| = \left|\left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0\right\rangle \times \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0\right\rangle\right| \Delta u \Delta v$$

$$= \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} \right\| \Delta u \Delta v = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k} \right| \Delta u \Delta v$$

$$= \text{abs val}\left(\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \right) \Delta u \Delta v$$

$$= \text{abs val}\left(\frac{\partial(x,y)}{\partial(u,v)} \right) \Delta u \Delta v$$

in the limit as $\Delta u, \Delta v \to 0$, the approximation by the parallelogram becomes better,

$$dx\,dy = \text{abs. val.}\left(\frac{\partial(x,y)}{\partial(u,v)} \right) du\,dv$$

$$dx\,dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$\left|\text{Jacobian}\right| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$ measures infinitesimal distortion of areas.

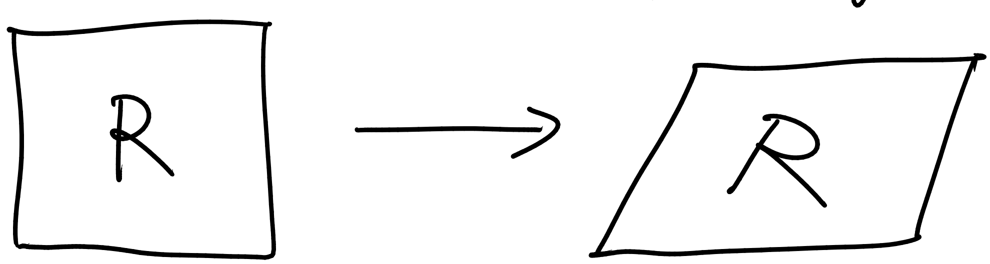
Why do we take absolute value of Jacobian?
Because Jacobian could be negative, and we don't want negative areas.

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} < 0$$

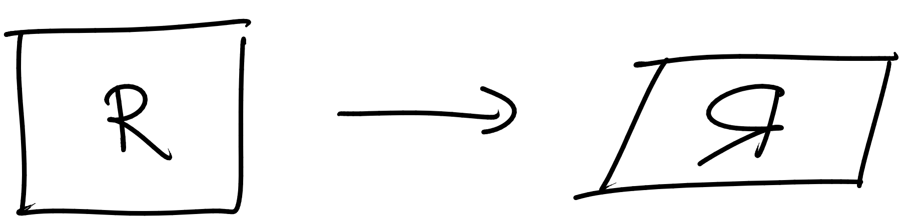
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1 \quad \text{area is preserved.}$$

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Jacobian >0 orientation preserving



Jacobian <0 orientation reversing

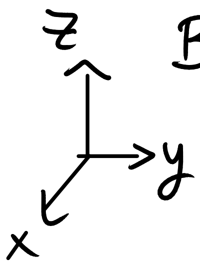


Examp $x = u$, $y = -v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$$

Triple integrals $\iiint_E f(x,y,z)\,dV$

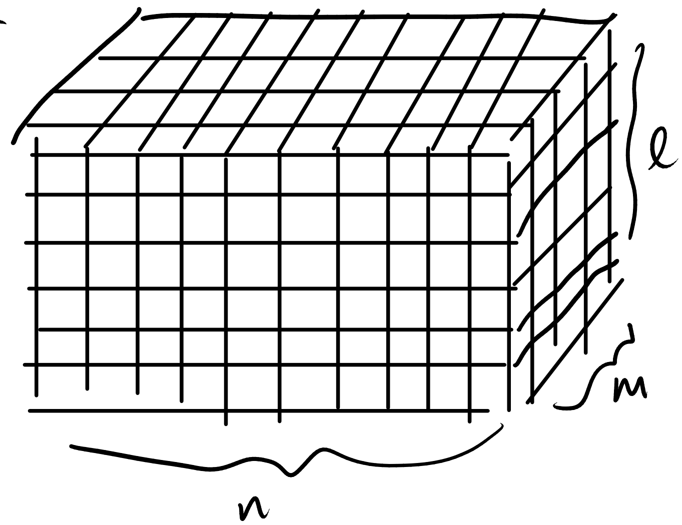
E = 3d region $dV = dx\,dy\,dz$



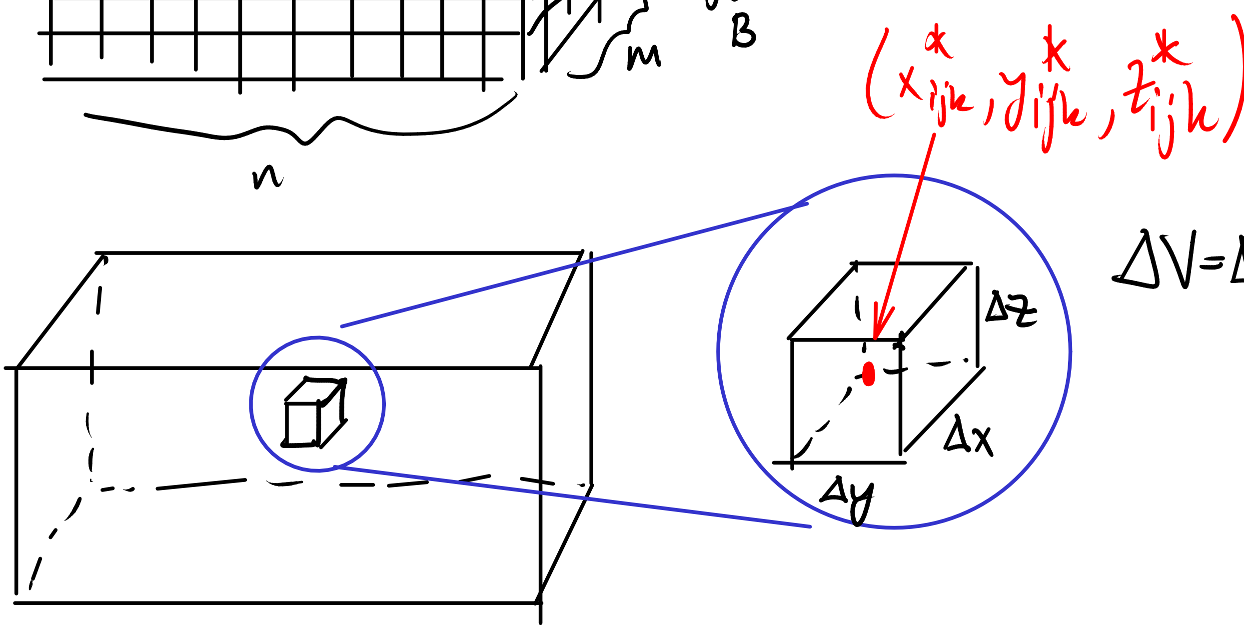
B = Rectangular Box

$B = \{a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$

$= [a,b] \times [c,d] \times [r,s]$



$$\iiint_B f(x,y,z)\,dV$$



$\Delta V = \Delta x \Delta y \Delta z$

Riemann sum

$$\iiint_B f(x,y,z)\,dV \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{\ell} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V$$

To get true value of integral: Take limit $n, m, \ell \to \infty$

$\Delta x, \Delta y, \Delta z \to 0$

Can do this as iterated integral

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

$$B = \{a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$$

Read integrals from inside→out

There are 6 possible orders

$dx\,dy\,dz$ $dy\,dx\,dz$ $dz\,dx\,dy$

$dx\,dz\,dy$ $dy\,dz\,dx$ $dz\,dy\,dx$

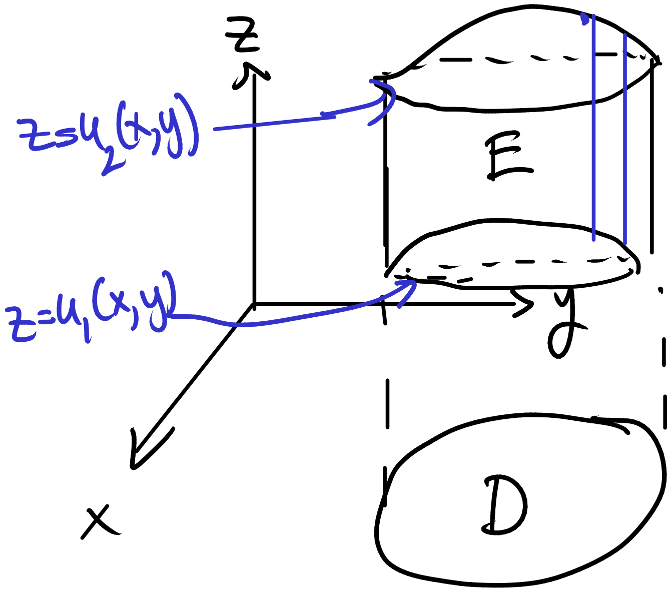
$$\int_0^1\int_0^1\int_0^1 xyz^2\,dx\,dy\,dz = \int_0^1\int_0^1 \left[\frac{x^2}{2} y z^2\right]_{x=0}^1 dy\,dz$$

$$= \int_0^1\int_0^1 \frac{1}{2} y z^2\,dy z = \int_0^1 \left[\frac{1}{2}\frac{y^2}{2} z^2\right]_{y=0}^1 dz = \int_0^1 \frac{1}{2}\cdot\frac{1}{2} z^2 dz$$

$$= \frac{1}{4}\left[\frac{z^3}{3}\right]_0^1 = \frac{1}{4}\cdot\frac{1}{3} = \frac{1}{12}$$

More general region: Type 1 in 3d

Bounded by two graphs of functions of (x,y).



3d region E sits over a 2d-region D in the (x,y)-plane

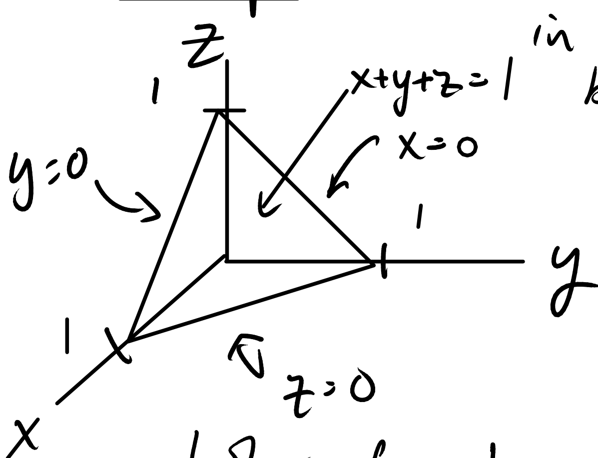
Bounded by 2 graphs

Integrate with respect to z first. then integrate over D.

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

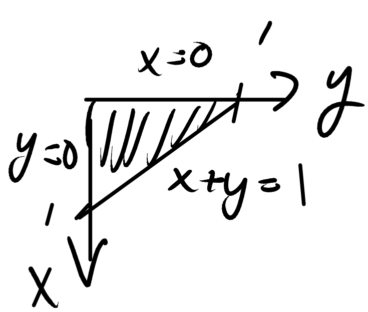
then deal with integral over D as usual.

<u>Example</u> E = tetrahedron in first octant bounded by



$$\iiint_E z\,dV$$

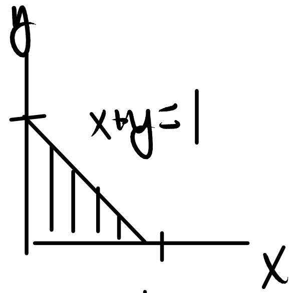
looking from above (along z-axis)



this is D.

Bottom $z=0$ Top = slanted face $z=1-x-y$

$$\iiint_E z\,dV = \iint_D \left[\int_0^{1-x-y} z\,dz\right] dA$$



$$\iint_D [\;]\,dA = \int_0^1 \int_0^{1-x} [\quad]\,dy\,dx$$

$$\iiint_E z\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx = \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\,dy\,dx$$

$$= \int_0^1 \left[-\frac{(1-x-y)^3}{6}\right]_0^{1-x} dx = \int_0^1 +\frac{1}{6}(1-x)^3\,dx$$

$$= \left[-\frac{1}{24}(1-x)^4\right]_0^1 = +\frac{1}{24}$$