Change of variables in double integrals (15.10)

(Converting between different coordinate systems)

1 var $$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

$x = g(u)$ $\qquad\qquad$ $a = g(c)$

$dx = g'(u)\,du$ $\qquad\qquad$ $b = g(d)$

g is the transformation from the u-coordinate to the x-coordinate

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Polar coordinates transformation $(r,\theta) \to (x,y)$
$$\begin{cases} x = r\cos\theta = g(r,\theta) \\ y = r\sin\theta = h(r,\theta) \end{cases}$$

$(x,y) = T(r,\theta)$

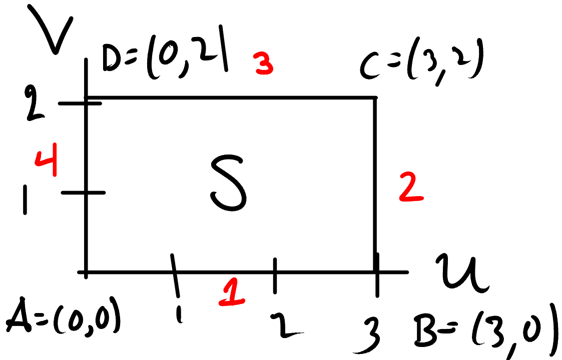
Generally: consider two coordinate systems (x,y) and (u,v)

with a <u>Transformation</u> between them

$$(x,y) = T(u,v) \qquad \underbrace{\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}}_{\text{components of } T}$$

Transformations <u>map</u> regions in (u,v) coords to regions in (x,y) coords

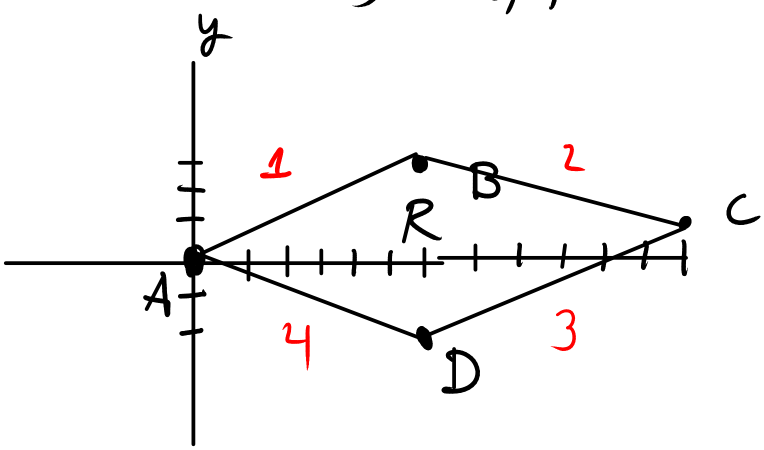
$S = \{0 \le u \le 3,\ 0 \le v \le 2\}$ $\qquad T: \begin{cases} x = 2u + 3v \\ y = u - v \end{cases}$



Side $v = 0$ $\quad 0 \le u \le 3$

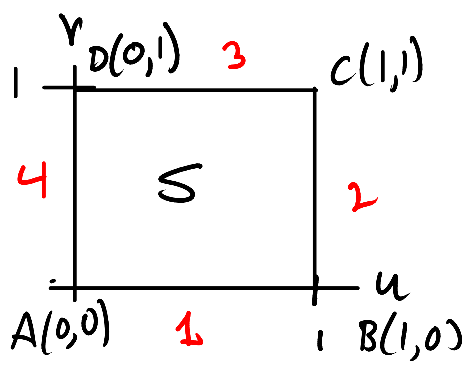
$\begin{cases} x = 2u & 0 \le u \le 3 \\ y = u & 0 \le u \le 3 \end{cases}$

$\hookrightarrow x = 2y \Rightarrow y = \frac{1}{2}x$



R = parallelogram is the <u>image</u> of S under the transformation T.

$S = \{0 \le u \le 1,\ 0 \le v \le 1\}$ $\qquad T: \begin{cases} x = v \\ y = u(1+v^2) \end{cases}$



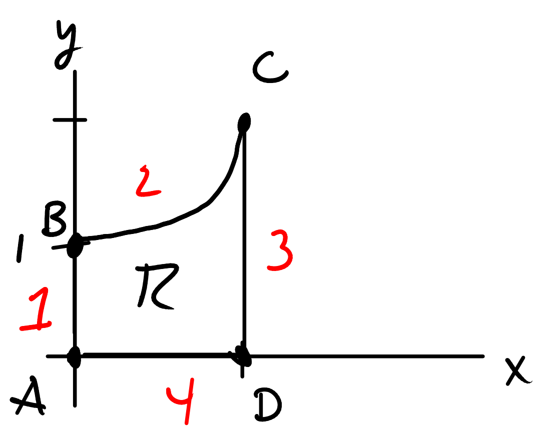


Image of side 2:

$u = 1 \quad 0 \le v \le 1$

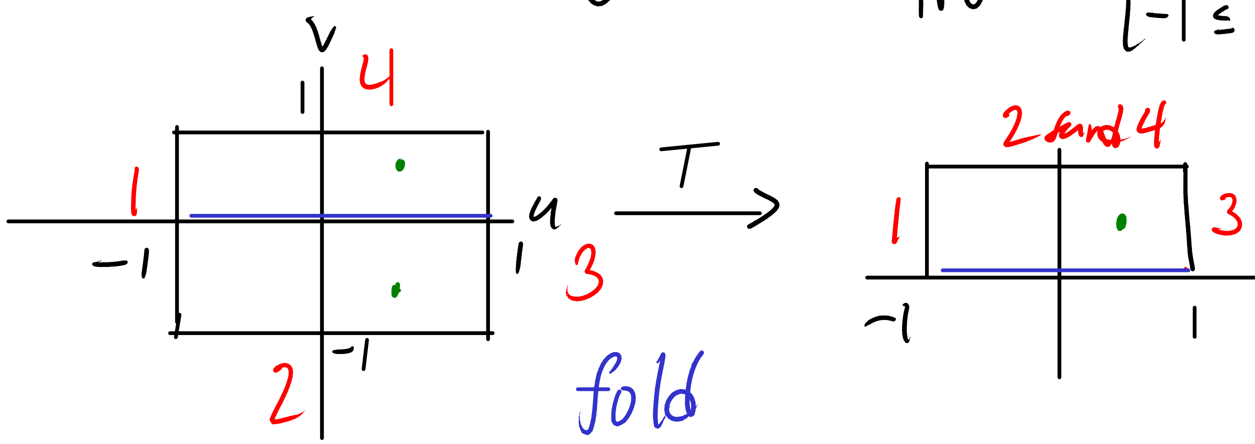
$x = v$

$y = 1(1+v^2) \Rightarrow y = 1 + x^2$

Linear transformations (all variables appear to first power only) map straight lines to straight lines.

In general a transformation $T(u,v) = (x,y)$ might not be invertible

$$T(u,v) = (x=u,\ y=v^2) \quad \text{apply to} \quad \left\{ \begin{array}{l} -1 \le u \le 1 \\ -1 \le v \le 1 \end{array} \right\}$$



This transformation is not one-to-one so it's not a valid change of coordinates.

If a transformation is one-to-one, it's invertible

$$T^{-1}(x,y) = (u,v)$$

given by solving $\begin{array}{l} x = g(u,v) \\ y = h(u,v) \end{array}$ for u and v.

Change of variables in double integrals

Transformation

$x = g(u,v)$ $\longrightarrow$ Matrix of partial derivatives

$y = h(u,v)$

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} g_u & h_u \\ g_v & h_v \end{bmatrix}$$

Jacobian of the transformation = determinant of matrix of partial derivatives

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

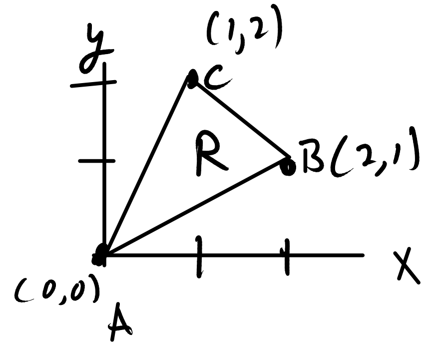
$$\iint_P f(x,y)\,dA = \iint_S f(g(u,v),h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

absolute value of Jacobian

$$\bullet\ dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Jacobian measures how much the transformation T distorts areas.

$$\iint_R (x-3y)\,dA$$



use $x = 2u+v$

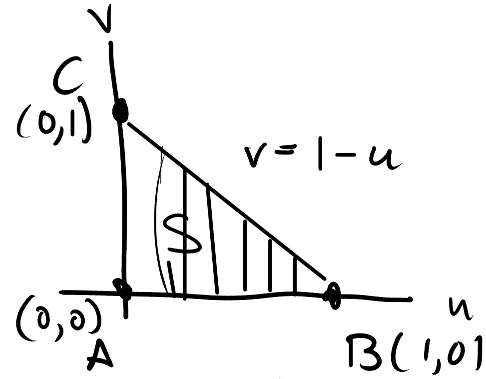
$y = u+2v$

Inverse $\quad x-2y = 2u+v-2u-4v = -3v$

$y-2x = u+2v-4u-2v = -3u$

$u = -\frac{1}{3}(y-2x)$

$v = -\frac{1}{3}(x-2y)$



Jacobian:

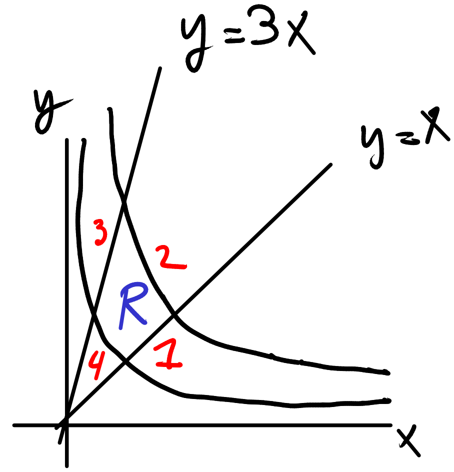
$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2\cdot 2-1\cdot 1 = 3$$

$$\iint_S (2u+v-3(u+2v))\ 3\,du\,dv$$

$$= \iint_S (-u-5v)(3)\,du\,dv$$

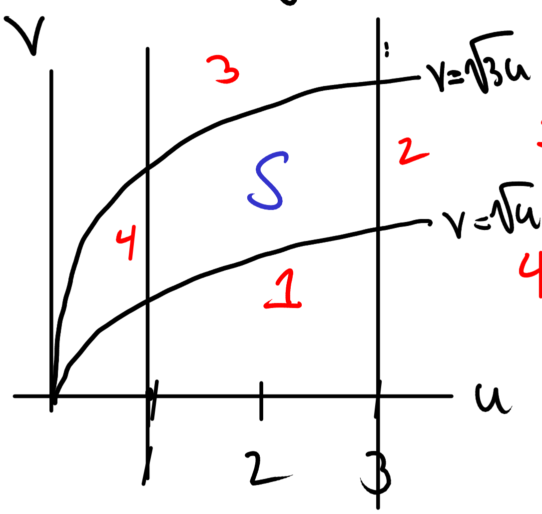
$$= \int_0^1\int_0^{1-u} (-u-5v)(3)\,dv\,du$$

$$\iint_R xy\,dA \qquad R \text{ in first quadrant bounded by } \begin{array}{l} y = x \\ y = 3x \\ xy = 1 \\ xy = 3 \end{array}$$



use $\begin{cases} x = u/v \\ y = v \end{cases}$

$\begin{cases} u = xy \\ v = y \end{cases}$



1	$y = x$	$v = u/v$ $v^2 = u$
2	$xy = 3$	$u = 3$
3	$y = 3x$	$v = 3u/v$ $u = \frac{1}{3}v^2$
4	$xy = 1$	$u = 1$

$$\iint_R xy\,dA = \iint_S u \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$x = u/v$
$y = v$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/v & 0 \\ -\frac{u}{v^2} & 1 \end{vmatrix} = 1/v$$

$$\iint_S u \frac{1}{v}\,du\,dv = \int_{u=1}^{u=3} \int_{v=\sqrt{u}}^{v=\sqrt{3u}} \frac{u}{v}\,dv\,du$$

$$\iint_R \frac{x-2y}{3x-y}\, dA \qquad R \text{ bounded by}$$

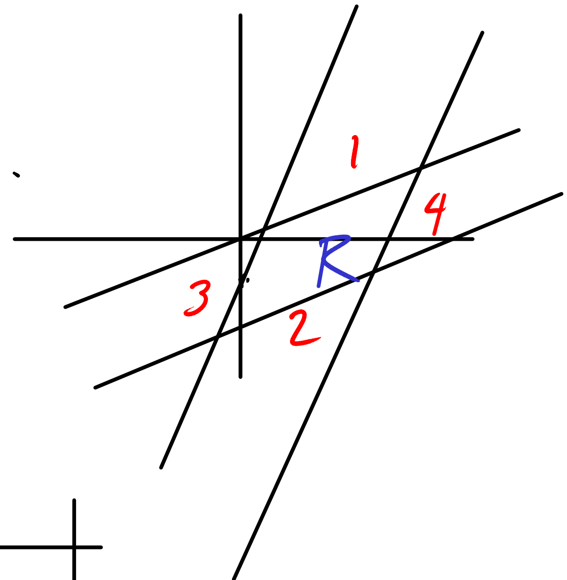
$1:\ x-2y=0 \qquad y=\frac{1}{2}x$

$2:\ x-2y=4 \qquad y=\frac{1}{2}x-2$

$3:\ 3x-y=1 \qquad y=3x-1$

$4:\ 3x-y=8 \qquad y=3x-8$

Compute the integral using an appropriate change of variables.



$\downarrow$

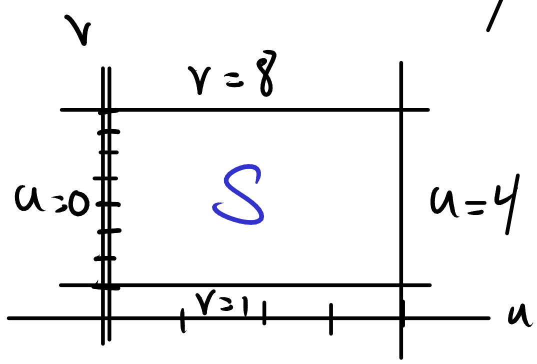
$1:\ u=0$

$2:\ u=4$

$3:\ v=1$

$4:\ v=8$

Try: $\begin{cases} u = x-2y \\ v = 3x-y \end{cases}$



For: $\dfrac{\partial(x,y)}{\partial(u,v)}$ need to write $\begin{matrix} x=g(u,v) \\ y=h(u,v). \end{matrix}$